${}^{p}M_n$ perfect numbers

Let's introduce a research paper whose title is " ${}^{p}M_n$ perfect numbers". Here p is an odd prime number and ${}^{p}M_n$ is denoted by $1+p+\cdots+p^{n-1}$ and it is called a p- Mersenne number. ${}^{p}M_n$ perfect numbers is defined by Shogo KIRIYAMA who is a member of the Math club and a student of author's class in 2017. A positive integer a is called a perfect number when a is equal to the sum of its proper positive divisors. It is well known theorem that a is an even perfect number if and only if $a=2^{n-1}\times M_n$ where $M_n=1+2+\cdots+2^{n-1}$ and is a prime number. This theorem was found by Euclid and was proved by Euler. So, Kiriyama asked a question which is "Which properties do the numbers $p^{n-1}\times {}^{p}M_n$ have?" He started his mathematical research to solve this question and could define ${}^{p}M_n$ perfect numbers.

A perfect number is a positive integer which is equal to the sum of its proper positive divisors. The first perfect number is 6. Its proper divisors are $1,2,$ and 3, and $1+2+3=6$. The next perfect number is $28=1+2+4+7+14$. This is followed by the perfect numbers $496,$ 8128, and 33550336. On the other hand, the following theorem for even perfect numbers is well known.

Theorem (Euclid-Euler). a is an even perfect number if and only if $a=2^{n-1}\times M_n$ and M_n a prime number.

The following are unsolved problems.

"*Are there infinitely many perfect numbers?*"

"*Are there any odd perfect numbers?*"

Most students said the answers to these problems will be "yes". Most students said that we needed a supercomputer to research the above problems. A student made an interesting remark.

> *How about a prime number $1+3+\cdots+3^{n-1}$? For example, the prime number 13 equals to $1+3+3^2$ and the proper positive divisors of the number $117=3^2\times 13$ are $1,3,9,13,26$. But,*
>
> $$117\neq 1+3+9+13+26.$$
>
> *Hence, 117 is not a perfect number.*

This research is by S. Kiriyama (cf. pp181-190 in [2]). Let p be a prime number and ${}^{p}M_n$ be a p-Mersenne prime number which is defined by ${}^{p}M_n=1+p+\cdots+p^{n-1}$ is prime. First he found the following formula

$$p^{n-1}\times {}^{p}M_n=(p-1)\tau\left(p^{n-1}\times {}^{p}M_n\right)-(p-2)\,{}^{p}M_n$$

where $\tau(x)$ is the sum of proper positive divisors of x. From this, he had an idea of new perfect number.

Definition 3. A natural number a is called ${}^{p}M_n$ perfect number when a satisfies the condition

$$a=(p-1)\tau(a)-(p-2)\,{}^{p}M_n.$$

Kiriyama researched many ${}^{p}M_n$ perfect numbers by using a computer and he found a important number ${}^{p}K_{d,n}$ which was defined by $(p-1)\,{}^{p}M_d-(p-2)\,{}^{p}M_n$ where $d\geq n$. In particular, when $p=2$, we have ${}^{2}K_{d,n}={}^{2}M_d$ which is a Mersenne number. He proved that if ${}^{p}K_{d,n}$ is a prime number then $p^{n-1}\times {}^{p}K_{d,n}$ is a ${}^{p}M_n$ perfect number. So, Kiriyama made the following problem.

Problem 1. Can every ${}^{p}M_n$ perfect number be represented by the from $p^{n-1}\times {}^{p}K_{d,n}$, where ${}^{p}K_{d,n}$ is a prime number?

When $p=2$ Problem 1 becomes the Theorem (Euclid-Euler). Kiriyama and the author think that to prove Problem 1 is difficult in the case $p\geq 3$. He proved the following theorem.

Theorem 2. Assume that $p\geq 3$ and $d=cn$ where c is a positive integer. Then Problem 1 is correct.

Reference

[2] Matsuda,O., and Kurata, H., and Horihata, Y., and Koshimizu, H., Research Reports of Mathematics by Kosen Students 4, Japan Society of Mathematical Education, Division for Colleges and Universities, Vol.23, No.1, 149-190 (2017) , Japanese.