Pythagorean triples on M_3

Let's introduce a research paper whose title is " Pythagorean triples on M_3". Here M_3 is the subring of the matrix ring with degree 2, i.e.,

$$M_3 = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \text{ are non negative integers.}\right\}$$

This was researched by Yoshifumi YABE who was a student of author's class in 2015, as well. He defined the Pythagorean triple (A, B, C) on M_3 by $A^2 + B^2 = C^2$. Furthermore, he asked a question, which is "Can you define primitive Pythagorean triples on M_3?" He started his mathematics research to solve this question.

This research is by Y. Yabe (cf. pp167-180 in [2]). Let $A = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix}$ and $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix}$. A triple (A, B, C) is called the Pythagorean triple on M_3 when (A, B, C) satisfies the two conditions (1) $A^2 + B^2 = C^2$, (2) numbers except for a_2, b_2 and c_2 are positive. For example when

$$A = \begin{pmatrix} 9 & 1 \\ 0 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & 3 \\ 0 & 12 \end{pmatrix}, \quad C = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}$$

(A, B, C) is a Pythagorean triples. In this example, putting $A' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}, B' = \begin{pmatrix} 4 & 3 \\ 0 & 4 \end{pmatrix}, C' = \begin{pmatrix} 5 & 3 \\ 0 & 5 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$A = LA'R,\ B = LB'R, C = LC'R \text{ and } A'^2 + B'^2 = C'^2.$$

From this case, Yabe defined a reducible (A, B, C) as follows.

Definition 1. *(A, B, C) is called reducible when (A, B, C) satisfies the three conditions (1) there exist elements $A', B', C', L,$ and R of M_3 such that $A = LA'R, B = LB'R$ and $C = LC'R$, (2) L and R don't have invertible matrixes, (3) $A'^2 + B'^2 = C'^2$. Moreover, (A, B, C) is called irreducible when (A, B, C) is not reducible.*

Yabe thought if triples by numbers (a_1, b_1, c_1) and (a_3, b_3, c_3) were primitives then (A, B, C) could be irreducible. However he found an example such that (a_3, b_3, c_3) is not primitive and (A, B, C) is irreducible. Namely,

$$A = \begin{pmatrix} 7 & 5 \\ 0 & 16 \end{pmatrix}, \quad B = \begin{pmatrix} 24 & 23 \\ 0 & 30 \end{pmatrix}, \quad C = \begin{pmatrix} 25 & 23 \\ 0 & 34 \end{pmatrix}.$$

Let $\wp$ be the set of Pythagorean triples on M_3. Yabe defined homomorphic map $f_h : M_3 \to M_3$ by $f_h(A) = \begin{pmatrix} a_1 & ha_2 \\ 0 & a_3 \end{pmatrix}$ for any integer h. Moreover, for any two triples (A, B, C) and (A', B', C') in $\wp$, he defined the relation $\sim$ on $\wp$ by $(A, B, C) \sim (A', B', C') \Leftrightarrow$ there exists a integer h such that $f_h(A) = A', f_h(B) = B'$ and $f_h(C) = C'$. Then the relation $\sim$ becomes the equivalence relation. From this, we have a quotient set $\wp/\sim$. A triple (A, B, C) with $\gcd(a_2, b_2, c_2) = 1$ is important because it be a representative of the equivalence class $[(A, B, C)]$. So Yabe defined $\wp_{h=1}$ to be the set of (A, B, C) with $\gcd(a_2, b_2, c_2) = 1$. Let $A_h = f_h(A)$, $B_h = f_h(B)$ and $C_h = f_h(C)$.

Definition 2. *(A, B, C) is reducible in $\wp_{h=1}$ when (A, B, C) satisfies the three conditions (1) there exist an integer h and elements L and R of M_3 such that, $A = LA_hR, B = LB_hR$ and $C = LC_hR$, (2) L and R don't have invertible matrixes, (3) $A_h^2 + B_h^2 = C_h^2$. Moreover, (A, B, C) is irreducible in $\wp_{h=1}$ when (A, B, C) is not reducible in $\wp_{h=1}$.*

Yabe proved the following theorem.

Theorem 1. Let (A, B, C) be a Pythagorean triple in $\wp_{h=1}$. Then (A, B, C) is irreducible in $\wp_{h=1}$. if and only if (a_1, b_1, c_1) and (a_3, b_3, c_3) are primitive.

Reference

[2] Matsuda,O., and Kurata, H., and Horihata, Y., and Koshimizu, H., Research Reports of Mathematics by Kosen Students 4, Japan Society of Mathematical Education, Division for Colleges and Universities, Vol.23, No.1, 149-190 (2017) , Japanese