Pascal zeta functions

Let's introduce a research paper whose title is "Pascal zeta functions". Pascal zeta functions was defined by Huga NAKANO, who was a student of author's class in 2015. The first line of Pascal triangle is the sequence $\{1, 1, 1, \cdots\}$. The 2nd line is the sequence $\{1,2,3, \cdots\}$ by natural numbers. The 3rd line is the sequence $\{1, 3,6,10, \cdots\}$ by triangle numbers. Nakano imaged the Riemann Zeta function $\zeta(s) = 1^s + 2^s + 3^s + 4^s + \cdots$ by the sequence of the 2nd line. Moreover, he considered the function $P_3(s) = 1^s + 3^s + 6^s + 10^s + \cdots$ by the sequence of the 3rd line. In general, he defined the n-th Pascal zeta function $P_n(s)$ by the sequence of the n-th line of Pascal triangle.

This research is by H. Nakano (cf. pp149-153 in [1]). Let $\{{}^N a_n\}$ be the sequence of the N-th line of Pascal triangle and let $P_N(s) = \sum_{n=1}^{\infty}({}^N a_n)^{-s}$. Let's see the way of the calculation of $P_2(s)$ by Nakano.

$$({}^2 a_n)^{-s} = \frac{n^{-s}(n+1)^{-s}}{2^{-s}} = 2^s n^{-s} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-s-k} = 2^s \sum_{k=0}^{\infty} \binom{-s}{k} n^{-2s-k}.$$

So, we have

$$P_2(s) = \sum_{n=1}^{\infty} ({}^2 a_n)^{-s} = 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} n^{-2s-k}.$$

In the above equality, whether or not $\Sigma_{n=1}^{\infty}$ and $\Sigma_{k=0}^{\infty}$ can be switched is a problem. But, he continued the calculation by assuming that this operation was correct. As a result, the equality

$$P_2(s) = 2^s \sum_{k=0}^{\infty} \binom{-s}{k} \zeta(2s+k) \tag{4.1}$$

was obtained. From this, we have

$$\text{“}1 + 3 + 6 + 10 + \cdots\text{"} = P_2(-1) = 2^{-1}\left\{\binom{1}{0}\zeta(-1) + \binom{1}{1}\zeta(0)\right\} = -\frac{1}{24},$$

$$\text{“}1^2 + 3^2 + 6^2 + 10^2 + \cdots\text{"} = P_2(-2) = 2^{-2}\left\{\binom{2}{0}\zeta(-4) + \binom{2}{1}\zeta(-3) + \binom{2}{2}\zeta(-2)\right\} = \frac{1}{240}.$$

After that, he studied the Euler-Maclaurin formula, and so he tried to calculate $P_2(s)$ by using the formula. Also he had $P_2(-1) = -1/24$ and $P_2(-2) = 1/240$. He and the author think that above formula (4.1) gotten by him is correct.

In general, he considered the formula of $P_n(s)$.

$$P_n(s) = (n!)^s \sum_{k_1,k_2,\dots,k_{n-1}=0}^{\infty} \binom{-s}{k_1}\binom{-s-k_1}{k_2}\cdots\binom{-s-k_1-\dots-k_{n-2}}{k_{n-1}} \begin{bmatrix} n \\ 1 \end{bmatrix}^{k_1} \begin{bmatrix} n \\ 2 \end{bmatrix}^{k_2} \cdots \begin{bmatrix} n \\ n \end{bmatrix}^{k_{n-1}} \zeta(ns + (n-1)k_1 + \dots + k_{n-1})$$

where $\binom{n}{k}$ is the number of possible combinations of k objects from a set of n objects, and $\begin{bmatrix} n \\ k \end{bmatrix}$ is the Stirling number of the first kind by n and k. In particular, he had

$$P_n(-1) = -\frac{1}{n!} \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \zeta(-k).$$

He likes the above expression very much. Moreover, he studied an integral expression of $P_2(s)$. So he proved the following formula

$$P_2(s) = \frac{1}{2\Gamma(s)} \int_0^{\infty} \left(\sqrt[8]{e^x}\theta_2(0, e^{-x/2}) x^{s-1}\right) dx$$

where $\Gamma(s)$ is the gamma function and . $\theta_2(*,*)$ is Jacobi theta function of the second kind. But, he and the author do not know the way of analytic continuation of the function $P_2(s)$, yet.

Reference

[1] Matsuda,O., and Kurata, H., and Saito, J., Research Reports of Mathematics by Kosen Students 3, Japan Society of Mathematical Education, Division for Colleges and Universities, Vol.22, No.1, 125-160 (2016), Japanese