

The punctual Hilbert schemes of degree two for monomial plane curve singularities[†]

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In this paper, we show that punctual Hilbert schemes of degree two for monomial plane curve singularities are isomorphic to a projective line.

Key words: Hilbert schemes of points, irreducible curve singularity

1. Introduction

Let k be an algebraically closed field of characteristic zero. In this paper, we consider a monomial plane curve singularity given by

$$(1) \quad x = t^a, \ y = t^b \quad (a < b, \gcd(a, b) = 1).$$

Its local ring $\mathcal{O}$ is isomorphic to $k[[t^a, t^b]]$. Let δ be the δ -invariant of $\mathcal{O}$. In our case, we have $\delta = (a-1)(b-1)/2$. In 6), Pfister and Steenbrink defined a special subset of the Grassmannian $\mathrm{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ for a given monomial plane curve singularity. We call it the Pfister-Steenbrink variety (PS variety) for the given singularity. They showed the existence of the punctual Hilbert scheme of degree r which parametrizes the ideals of codimension r in $\mathcal{O}$. It is realized as a connected component of the PS variety.

Pfister and Steenbrink analyzed the structure of the PS varieties for certain curve singularities. The first author argued the rationality of punctual Hilbert schemes for monomial plane curve singularities in 11). The punctual Hilbert schemes for monomial plane curve singularities of types A_{2d} , E_6 and E_8 were also studied in 9) and 10). On the other hand, the punctual Hilbert schemes for nodal singularity were considered in 7). The structure of the punctual Hilbert schemes for cuspidal and nodal singularities

applied to the string theory in 5). Also refer to 1) and 3) about punctual Hilbert schemes. Our main theorem is stated as follows:

Theorem 1. *Let C be a monomial curve singularity given by the parametrization (1), the punctual Hilbert scheme of degree 2 for C is isomorphic to a projective line.*

In Section 2, we briefly recall Pfister-Steenbrink theory. In Section 3, we prove Theorem 1. Finally, we consider the case of the plane curve singularity defined by $x = t^4, y = t^5$ in Section 4.

2. Preliminaries

We recall Pfister-Steenbrink theory and prove some lemmas needed later. We fix notations. Consider a monomial plane curve singularity whose local ring is $\mathcal{O} = k[[t^a, t^b]]$. The notions explained in this section hold more general situations. For details, see 6). We denote by $\overline{\mathcal{O}}$ the normalization of $\mathcal{O}$. Namely, $\overline{\mathcal{O}} = k[[t]]$. We call the set $\Gamma := \{\mathrm{ord}(f) \mid f \in \mathcal{O}\}$ the *semi-group* of $\mathcal{O}$. An element of the set $G := \mathbb{N} \setminus \Gamma$ is called a gap of Γ . We may assume that the semigroup Γ of $\mathcal{O}$ is minimally generated by a, b . For $n \in \mathbb{N}$, set $\overline{I}(n) := \{\mathrm{ord}(f) \mid f \in \overline{\mathcal{O}}\}$ and $I(n) := \overline{I}(n) \cap \mathcal{O}$. We call $c := \min\{n \mid \overline{I}(n) \subset \mathcal{O}\}$ the *conductor* of Γ . Define the δ -invariant of $\mathcal{O}$ to be $\dim_k(\overline{\mathcal{O}}/\mathcal{O}) = \#G$. Then the relation $\delta + 1 \leq c \leq 2\delta$ holds. Furthermore, the following lemma is known. See 4) and 8).

Lemma 2 (Gorenstein). *For a plane curve singularity, we have $c = 2\delta$.*

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For $p, q \in \mathbb{N}$ ($p < q$), put $[p, q] := \{x \in \mathbb{N} \mid p \leq x \leq q\}$. For a nonzero ideal I in $\mathcal{O}$, we call $r := r(I) := \dim_k \mathcal{O}/I$ the *codimension* of I . We denote by $\Gamma(I) := \{\text{ord}(f) \mid f \in I\}$ the *order set* of I . Set $G(I) := \Gamma \setminus \Gamma(I)$ and $c(I) := \max\{n \in \Gamma(I) \mid n-1 \notin \Gamma(I)\}$.

Definition 3. An element J of $\text{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ is called *good*, if the multiplication $\mathcal{O} \times J \ni (f, \bar{v}) \mapsto \bar{f}\bar{v} \in J$ is defined and J is an $\mathcal{O}$ -submodule with respect to this multiplication. Set

$$\mathcal{M} := \{J \in \text{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta)) \mid J \text{ is good}\},$$

$$\mathcal{I}_r := \{I \mid I \text{ is an ideal of } \mathcal{O} \text{ with } \dim \mathcal{O}/I = r\}.$$

We prove some lemmas.

Lemma 4. An ideal I of $\mathcal{O}$ belongs to $\mathcal{I}_r$ if and only if we have $\sharp G(I) = r$.

Proof. We easily see that an ideal I belongs to $\mathcal{I}_r$ if and only if the relation

$$(2) \quad \mathcal{O}/I = \left\{ \sum_{i=0}^{r-1} a_i t^{d_i} \mid d_i \in G(I), d_i < d_{i+1} \right\}$$

holds. $\square$

Lemma 4 implies that the codimension of I depends on $\Gamma(I)$. For a Γ -module S , we denote by $\mathcal{I}(S)$ the set of all ideals in $\mathcal{O}$ with $\Gamma(I) = S$.

Proposition 5. There exists a finite number of distinct Γ -modules $S_1, \dots, S_h$ such that

$$(3) \quad \mathcal{I}_r = \bigcup_{i=1}^h \mathcal{I}(S_i)$$

where $\mathcal{I}(S_i) \cap \mathcal{I}(S_j) = \emptyset$ for $i \neq j$.

Proof. For a given codimension r , there exists a finite number of sets of gaps which satisfy the condition (2). So there exists finitely many distinct Γ -modules $S_1, \dots, S_h$ which are the order sets of ideals in $\mathcal{I}_r$. We obtain the desired decomposition (3). It is clear that $\mathcal{I}(S_i) \neq \mathcal{I}(S_j)$ for $i \neq j$. $\square$

Remark 6. An algorithm to compute Γ -modules $S_1, \dots, S_h$ was given in [12].

We consider the following composition map:

$$\psi : \mathcal{M} \rightarrow \text{Gr}(\delta, 2\delta) \rightarrow \text{M}_{\delta, 2\delta}(k) / \sim \rightarrow \mathbb{P}^N$$

Put $J = \langle f_1, \dots, f_\delta \rangle_k$ where $f_i = \sum_{j=0}^{2\delta-1} a_{ij} t^j \in \overline{\mathcal{O}}/I(2\delta)$. We identify f_i with a point $\mathbf{a}_i = (a_{i0}, \dots, a_{i, 2\delta-1})$ of $k^{2\delta}$. The first map in ψ is given by this identification. Let A_J be a $\delta \times 2\delta$ matrix whose i -th row is $\mathbf{a}_i$. We call A_J the *reperesent matrix* of J . The second map in ψ sends a k -vector space $\langle \mathbf{a}_1, \dots, \mathbf{a}_\delta \rangle_k$ to the coset $\overline{A_J}$ of A_J . Here the equivalence relation $\sim$ is the similarity of matrices. We may assume that $\overline{A_J}$ is represented by the reduced row echelon form. The third map in ψ is Plücker embedding with $N = \binom{2\delta}{\delta} - 1$. For $r > 0$, Pfister and Steenbrink defined a map $\varphi_r : \mathcal{I}_r \rightarrow \mathcal{M}$ by $\varphi_r(I) = t^{-r} I / I(2\delta)$. For φ_r , the following proposition was shown in [6]:

Proposition 7. The map φ_r has the following properties.

- (i) The map φ_r is injective for any r .
- (ii) The map φ_r is bijective for $r \geq 2\delta$.
- (iii) The set $\varphi_r(\mathcal{I}_r)$ is a Zariski closed set in $\mathcal{M}$.

Definition 8. For a given monomial plane curve singularity, we call $\mathcal{M}$ and $\mathcal{M}_r := \varphi_r(\mathcal{I}_r)$ the *Pfister-Steenbrink variety* (PS variety) and the *punctual Hilbert scheme of degree r* respectively.

The following fact follows from (ii) in Proposition 7.

Corollary 9. The punctual Hilbert scheme $\mathcal{M}_r$ with $r \geq 2\delta$ coincides with the PS variety $\mathcal{M}$.

We define *Schubert cell* $W_{a_1, \dots, a_\delta}$ for $\delta \geq a_1 \geq \dots \geq a_\delta \geq 0$ to be the set of all elements W in $\text{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ which satisfy $\dim(W \cap V_{\delta+i-a_i}) = i$ for $1 \leq i \leq \delta$ and $\dim(W \cap V_j) < i$ for $j < \delta + i - a_i$. The 2δ -dimensional k -vector space $\overline{\mathcal{O}}/I(2\delta)$ has the canonical flag

$$0 \subset V_1 \subset V_2 \subset \dots \subset V_{2\delta} = \overline{\mathcal{O}}/I(2\delta)$$

where $V_i = \overline{I}(2\delta - i) / I(2\delta)$ for $1 \leq i \leq 2\delta$. This induces a partition of $\text{Gr}(\delta, \overline{\mathcal{O}}/I(2\delta))$ into Schubert cells $W_{a_1, \dots, a_\delta}$. For an index set $\Lambda = \{a_1, \dots, a_\delta\}$, we sometimes write W_Λ instead of $W_{a_1, \dots, a_\delta}$.

Proposition 10. For the Schubert cells, we have $W_{b_1, \dots, b_\delta} \subset \overline{W_{a_1, \dots, a_\delta}}$ if and only if $b_i \geq a_i$ holds for $1 \leq i \leq \delta$.

For the details about Schubert cells, see 2).

Let Δ be a subset of $[0, 2\delta - 1]$ such that $\sharp\Delta = \delta$ and $\Delta \cup [2\delta, \infty)$ is a Γ -module. Then we define $\mathcal{M}_r(\Delta)$ to be the subset of $\mathcal{M}_r$ parametrizing ideals I with $\Delta = (\Gamma(I) - r) \cap [0, 2\delta - 1]$.

Lemma 11. *Put $\Delta = \{b_1, \dots, b_\delta\}$ where $0 \leq b_1 < \dots < b_\delta < 2\delta$. Setting*

$$(4) \quad a_{\delta-i+1} = b_i - i + 1 \text{ for } 1 \leq i \leq \delta,$$

we have $\mathcal{M}_r(\Delta) = \mathcal{M}_r \cap W_{a_1, \dots, a_\delta}$ for $r \in \mathbb{N}$.

Proof. Under the same conditions, the relation $\mathcal{M}_{2\delta}(\Delta) = \mathcal{M}_{2\delta} \cap W_{a_1, \dots, a_\delta}$ holds (see Lemma 5 in 1)). So we have

$$\mathcal{M}_r(\Delta) = \mathcal{M}_r(\Delta) \cap \mathcal{M}_{2\delta}(\Delta) = \mathcal{M}_r \cap (\mathcal{M}_{2\delta} \cap W_{a_1, \dots, a_\delta}) = \mathcal{M}_r \cap W_{a_1, \dots, a_\delta}. \quad \square$$

For a component $\mathcal{I}(S_i)$ of the decomposition (3), we have $\mathcal{M}_r(\Delta_i) = (\psi \circ \varphi_r)(\mathcal{I}(S_i))$ where $\Delta_i = (S_i - r) \cap [0, 2\delta - 1]$. The decompositions of $\mathcal{M}_r$ follows from Proposition 5.

Corollary 12. *Corresponding to (3), we have*

$$(5) \quad \mathcal{M}_r = \bigcup_{i=1}^h \mathcal{M}_r(\Delta_i).$$

Proposition 13. *Each component $\mathcal{M}_r(\Delta_i)$ in the decomposition (5) is isomorphic to an affine space whose dimension equals the number of coefficients in the generators of $\mathcal{M}_r(\Delta_i)$ (as $\mathcal{O}$ -modules).*

3. Proof of Theorem 1

In this section, we prove Theorem 1. Let C be a plane curve singularity given by (1). Put $\mathcal{O} = k[[t^a, t^b]]$. It follows from Lemma 2 that $2\delta = c = (a - 1)(b - 1)$. Write

$$\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{2\delta}, \dots\}$$

where $\gamma_i < \gamma_{i+1}$, $\gamma_1 = 0$, $\gamma_2 = a$, $\gamma_{2\delta} = (a - 1)(b - 1)$ and $\gamma_{2\delta+i} = (a - 1)(b - 1) + i$ for any $i \in \mathbb{N}$. By Lemma 4, we see that an ideal I of $\mathcal{O}$ belongs to $\mathcal{I}_2$ iff $\Gamma(I)$ coincides with one of Γ -modules $S_1 = \langle 2a, b \rangle$ and $S_2 = \langle a, b + a \rangle$. Here the notation $\langle p, q \rangle$ assigns the Γ -module generated by p and q . In our case, they are described as

$$\begin{aligned} S_1 &= \{\gamma_3, \gamma_4, \gamma_5, \gamma_6, \dots\} \quad (\gamma_3 = 2a), \\ S_2 &= \{\gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots\} \quad (\gamma_1 = a, \gamma_j = b). \end{aligned}$$

So the decomposition (3) in Proposition 5 is determined as

$$(6) \quad \mathcal{I}_2 = \mathcal{I}(S_1) \cup \mathcal{I}(S_2).$$

Furthermore, we easily see that

$$\begin{aligned} \mathcal{I}(S_1) &= \{(t^{2a}, t^b)\}, \\ (7) \quad \mathcal{I}(S_2) &= \{(t^a + pt^b, t^{b+a}) \mid p \in k\}. \end{aligned}$$

We obtain the decomposition (5) of $\mathcal{M}_2$ as

$$(8) \quad \mathcal{M}_2 = \mathcal{M}_2(\Delta_1) \cup \mathcal{M}_2(\Delta_2)$$

where

$$\begin{aligned} \Delta_1 &:= (S_1 - 2) \cap [0, 2\delta - 1] \\ &= \{\gamma_3 - 2, \gamma_4 - 2, \dots, \gamma_{2\delta+1} - 2, \gamma_{2\delta+2} - 2\}, \\ \Delta_2 &:= (S_2 - 2) \cap [0, 2\delta - 1] \\ &= \{\gamma_2 - 2, \dots, \gamma_{j-1} - 2, \gamma_{j+1} - 2, \\ &\quad \dots, \gamma_{2\delta+2} - 2\}. \end{aligned}$$

Here we rewrite Δ_1 and Δ_2 as

$$\begin{aligned} \Delta_1 &= \{m_1, \dots, m_{2\delta}\} \quad (m_i < m_{i+1}), \\ \Delta_2 &= \{n_1, \dots, n_{2\delta}\} \quad (n_i < n_{i+1}). \end{aligned}$$

According to (4), we define index set $\Lambda_1 = \{p_1, \dots, p_{2\delta}\}$ (resp. $\Lambda_2 = \{q_1, \dots, q_{2\delta}\}$) by $p_{\delta-i+1} = m_i - i + 1$ (resp. $q_{\delta-i+1} = n_i - i + 1$) for $1 \leq i \leq \delta$. It follows from (8), Lemma 11 and Proposition 10 that

$$\begin{aligned} \mathcal{M}_2 &= (\mathcal{M}_2 \cap W_{\Lambda_1}) \cup (\mathcal{M}_2 \cap W_{\Lambda_2}) \\ &= \mathcal{M}_2 \cap (\overline{W_{\Lambda_1}} \cup \overline{W_{\Lambda_2}}) = \mathcal{M}_2 \cap \overline{W_{\Lambda_2}} = \overline{\mathcal{M}_2(\Delta_2)}. \end{aligned}$$

Hence we have $\mathcal{M}_2 = \overline{W_2(\Delta_2)}$. It follows from Proposition 13 and (7) that $\mathcal{M}_2(\Delta_2) = \mathbb{A}_k^1$. Hence we conclude that $\mathcal{M}_2 = \mathbb{P}^1$. Theorem 1 has proved.

4. Example

In this section, we consider the plane curve singularity which given by $x = t^4$, $y = t^5$. We have $\mathcal{O} = k[[t^4, t^5]]$ and

$$\Gamma = \{0, 4, 5, 8, 9, 10, 12, 13, \dots\}$$

where $c = 12$. Putting

$$\begin{aligned} S_1 &= \{5, 8, 9, 10, 12, 13, \dots\}, \\ S_2 &= \{4, 8, 9, 10, 12, 13, \dots\}, \end{aligned}$$

we obtain the decomposition (6) of $\mathcal{I}_2$ and have

$$\begin{aligned}\mathcal{I}(S_1) &= \{(t^5, t^8)\}, \\ \mathcal{I}(S_2) &= \{(t^4 + pt^5, t^9) \mid p \in k\}.\end{aligned}$$

The images of these set by φ_r are

$$\begin{aligned}\mathcal{M}(\Delta_1) &= \{(t^3, t^6)/I(12)\}, \\ \mathcal{M}(\Delta_2) &= \{(t^2 + pt^3, t^7)/I(12) \mid p \in k\}.\end{aligned}$$

These are the components of $\mathcal{M}_2$ (see (8)). In this case, $\mathcal{M}_2$ is embedded in $\mathbb{P}^{923}$ by ψ . Let J_1 (resp. J_2) be an elements of $\mathcal{M}(\Delta_1)$ (resp. $\mathcal{M}(\Delta_2)$). Then their represent matrices are

$$\begin{aligned}M_{J_1} &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ M_{J_2} &= \begin{pmatrix} 0 & 0 & 1 & q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\end{aligned}$$

respectively. The images $\psi(J_1)$ and $\psi(J_2)$ are given in $\mathbb{P}^{923}$ by the following Plücker coordinates:

$$\begin{aligned}\psi(J_1) : \pi_\Lambda &= \begin{cases} 1 & \text{for } \Lambda_2 := \{4, 7, 8, 9, 11, 12\}, \\ 0 & \text{for the others,} \end{cases} \\ \psi(J_2) : \pi_\Lambda &= \begin{cases} 1 & \text{for } \Lambda_1 := \{3, 7, 8, 9, 11, 12\}, \\ q & \text{for } \Lambda_2 = \{4, 7, 8, 9, 11, 12\}, \\ 0 & \text{for the others.} \end{cases}\end{aligned}$$

Namely, they are represented by the vectors $\psi(J_1) = (0, \dots, 1, \dots, 0)$ ($\pi_2 = 1$),

$\psi(J_2) = (0, \dots, 1, \dots, q, \dots, 0)$ ($\pi_1 = 1, \pi_2 = q$). Here we rewrite π_{λ_1} and π_{λ_2} by π_1 and π_2 respectively. We infer from these facts that $\psi(\mathcal{M}_2)$ consists of all vectors $\mathbf{a} = (0, \dots, p, \dots, q, \dots, 0)$ with the conditions $\pi_1 = p$, $\pi_2 = q$ and $pq \neq 0$. Since ψ is injective, we identify $\psi(\mathcal{M}_2)$ with $\mathcal{M}_2$. Then there exists a natural isomorphism $\iota : \mathbb{P}^1 \rightarrow \mathcal{M}_2$ defined by $\iota((p : q)) = \mathbf{a}$.

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