Tsuyama-castle Function and Matrix Art

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The present paper constructs a class of complex-valued functions c-MagicMt$_\theta$ and associated operator theoretic tools, which serve as powerful tools for proving the Asymptotic Linearity Theorem Extension Conjecture (ALTEC) in the repeat space theory (RST). The real part of c-MagicMt$_\pi$ is what is called the Tsuyama-castle function. This function has an interesting inwardly repeating fractal structure and self-similarity, useful in the RST and in the Matrix Art Program of the Niagara Project, which is a special new part of the ongoing international, interdisciplinary, and inter-generational Second Generation Fukui Project.

Key words: the Fukui conjecture, repeat space theory (RST), Asymptotic Linearity Theorem Extension Conjecture (ALTEC), Banach spaces, Banach algebras

1. Introduction

In his later years, Kenichi Fukui (1918 - 1998, Nobel Prize 1981) presented several conjectures concerning the additivity problems of molecules having many identical moieties. Among them is what is called the ‘Fukui Conjecture on repeat sequences’ which has been playing a significant role in the development of the repeat space theory (RST) (cf. [1-20]), which is the central unifying theory in the first (cf. [1,2]) and the second (cf. [2,3]) generation Fukui Project.

The Asymptotic Linearity Theorem (ALT), which proves the Fukui conjecture in a broader context, plays a significant role in the RST. Proving the Asymptotic Linearity Theorem Extension Conjecture (ALTEC) reproduced in Section 3 (cf. [20] for details) is thus a fundamental problem in the repeat space theory. The present paper constructs a class of complex-valued functions c-MagicMt$_\theta$, and associated operator theoretic tools, which serve as powerful tools for proving the Asymptotic Linearity Theorem Extension Conjecture (ALTEC) in the repeat space theory (RST). The real part of c-MagicMt$_\pi$ is what is called the Tsuyama-castle function. This function has an interesting inwardly repeating fractal structure, useful in the RST and in the Matrix Art Program of the Niagara Project, which is a special new part of the ongoing international, interdisciplinary, and inter-generational Second Generation Fukui Project. Theory of Banach spaces and Banach algebras (cf. e.g. [21,22]) are fundamental for constructing the class of functions c-MagicMt$_\theta$.

2. Preparation

Throughout, let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote respectively the set of all positive integers, nonnegative integers, integers, real numbers, and complex numbers. We also need the following symbols for our purpose of constructing the above-mentioned class of functions c-MagicMt$_\theta$.

Let A be a nonempty set, let

$$A^2 := A \times A, \tag{2.1}$$

where $\times$ denotes the Cartesian product. Let $J := [0, 1] \subset \mathbb{R}$, thus we have

$$J^2 = J \times J \subset \mathbb{R}^2.$$

Let ∂J^2 denote the boundary of the set J^2 in $\mathbb{R}^2$ with the usual Euclidian metric.

Let $C(J^2, \mathbb{C})$ denote the complex Banach space of all complex-valued continuous functions on J^2 equipped with the norm given by

$$\|\varphi\| = \sup\{|\varphi(x)|: x \in J^2\}. \tag{2.2}$$

Let $C_0(J^2, \mathbb{C})$ denote the closed subspace of $C(J^2, \mathbb{C})$ defined by

$$C_0(J^2, \mathbb{C}) = \{\varphi \in C(J^2, \mathbb{C}): \varphi(x) = 0 \text{ for all } x \in \partial J^2\}. \tag{2.3}$$

(**Note**: $C_0(J^2, \mathbb{C})$ is a complex Banach space since it is a closed subspace of the complex Banach space $C(J^2, \mathbb{C})$.)

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Let

$C(\mathbb{R}^2, \mathbb{C}) := \{\varphi$: φ is a complex-valued continuous function defined on $\mathbb{R}^2\}$. (2.4)

Let

$C_0(\mathbb{R}^2, \mathbb{C}) := \{\varphi \in C(\mathbb{R}^2, \mathbb{C})$: $\varphi(x) = 0$ for all $x \in \partial J^2$ and $\varphi(x) = \varphi(x+y)$ for all $x \in J^2$ and $y \in \mathbb{Z}^2\}$. (2.5)

Let ^ denote the linear operator ^: $C_0(J^2, \mathbb{C}) \to C_0(\mathbb{R}^2, \mathbb{C})$ that sends φ to the unique element $\hat{\varphi} \in C_0(\mathbb{R}^2, \mathbb{C})$ with $\hat{\varphi}\big|_{J^2} = \varphi$, where $\hat{\varphi}\big|_{J^2}$ denotes the restriction of the function $\hat{\varphi}$ to the set J^2. Note that the linear mapping ^ is a bijection.

For each $n \in \mathbb{Z}^+$, define the linear operator $\hat{L}_n : C_0(\mathbb{R}^2, \mathbb{C}) \to C_0(\mathbb{R}^2, \mathbb{C})$ by

$$\hat{L}_n(\varphi)(x) = \frac{1}{n}(\varphi)(nx). \tag{2.6}$$

For each $n \in \mathbb{Z}^+$, define the bounded linear operator L_n: $C_0(J^2, \mathbb{C}) \to C_0(J^2, \mathbb{C})$ by

$$L_n(\varphi) = \hat{L}_n(\hat{\varphi})\big|_{J^2}. \tag{2.7}$$

Note that (2.6) and (2.7) imply that

$$\|L_n\| = \frac{1}{n}. \tag{2.8}$$

Let $\|\cdot\|_\infty$ denote the norm in $\mathbb{R}^2$ defined by

$$\|(x_1, x_2)\|_\infty = \mathrm{Max}\{|x_1|, |x_2|\}. \tag{2.9}$$

Let

$$\gamma := \frac{1}{2}(1,1) \in \mathbb{R}^2. \tag{2.10}$$

Let $\psi \in C_0(J^2, \mathbb{C})$ be the function defined by

$$\psi(x) = 1 - 2\|x-\gamma\|_\infty. \tag{2.11}$$

(**Note**: $\psi(x) = 0 \Leftrightarrow \|x-\gamma\|_\infty = \dfrac{1}{2} \Leftrightarrow x \in \partial J^2$.)

Let $\boldsymbol{B}(C_0(J^2, \mathbb{C})) := \boldsymbol{B}(C_0(J^2, \mathbb{C}), C_0(J^2, \mathbb{C}))$ denote the complex Banach algebra of all bounded linear operators acting on $C_0(J^2, \mathbb{C})$. For each $n \in \mathbb{Z}^+$ with $n \geq 2$, and $\theta \in \mathbb{R}$, define $\Omega_{2,n,\theta} \in \boldsymbol{B}(C_0(J^2, \mathbb{C}))$ by

$$\Omega_{2,n,\theta} = \sum_{k=0}^{\infty} e^{ik\theta} L_n^k, \tag{2.12}$$

where L_n^0 denotes the identity operator, and if $k \geq 1$, $L_n^k := L_n L_n \ldots L_n \in \boldsymbol{B}(C_0(J^2, \mathbb{C}))$, and where L_n is composed with itself k times. Note that

$$\|\Omega_{2,n,\theta}\| \leq \sum_{k=0}^{\infty} \|e^{ik\theta} L_n^k\|$$

$$\leq \sum_{k=0}^{\infty} \|L_n\|^k = \frac{1}{1-\frac{1}{n}}. \tag{2.13}$$

3. Magic Mountains

For each $\theta \in \mathbb{R}$, we can now define the complex-valued continuous function c-MagicMt$_\theta$: $[0, 1] \times [0, 1] \to \mathbb{C}$ by

$$\text{c-MagicMt}_\theta := \Omega_{2,2,\theta}(\psi). \tag{3.2}$$

For each $k \in \mathbb{Z}_0^+$, define $\mathrm{pyramid}_k \in C_0(J^2, \mathbb{C})$ by

$$\mathrm{pyramid}_k = L_2^k \psi. \tag{3.3}$$

Then, we may also define c-MagicMt$_\theta$ in terms of the pyramid$_k$ given above:

$$\text{c-MagicMt}_\theta = \sum_{k=0}^{\infty} e^{ik\theta}\mathrm{pyramid}_k. \tag{3.4}$$

(**Note**: The latter way of defining c-MagicMt$_\theta$ by (3.4) is suitable when one constructs computer graphics of the graph of c-MagicMt$_\theta$, whereas the former way of defining c-MagicMt$_\theta$ is mathematically more powerful especially when one utilizes c-MagicMt$_\theta$ for proving the ALTEC [20] reproduced below. The detailed development along this line of investigations shall be published elsewhere.)

For each $\theta \in \mathbb{R}$, let MagicMt$_\theta$: $J^2 \to \mathbb{R}$ denote the real part of the function c-MagicMt$_\theta$:

$$\mathrm{MagicMt}_\theta = \mathrm{Re}\Big(\sum_{k=0}^{\infty} e^{ik\theta}\mathrm{pyramid}_k\Big)$$

$$= \sum_{k=0}^{\infty}(\cos(k\theta))\mathrm{pyramid}_k\,. \qquad (3.5)$$

Now let us recall:

Asymptotic Linearity Theorem Extension Conjecture (ALTEC $C(I)$ version) *Let $a, b \in \mathbb{R}$ with $a < b$, and let $I = [a, b]$. The Asymptotic Linearity Theorem* (*ALT*) *cannot be extended from $AC(I)$ to $C(I)$, where $AC(I)$ denotes the functional space of all real valued absolutely continuous functions defined on the closed interval I, and $C(I)$ denotes the functional space of all real valued continuous functions defined on the closed interval I.*

This conjecture (ALTEC), which was referred to in Section 1, was first proved by the present author, and the second proof of this conjecture was obtained by him in a seminar called "Matrix Art Challenge Seminar" in Tsuyama National College of Technology (TNCT), in conjunction with the above defined continuous function with $\theta = \pi$: $\mathrm{MagicMt}_\pi : [0, 1] \times [0, 1] \to \mathbb{R}$ and the following Matrix Art picture of this function constructed by using MATLAB. The scale of the function has been changed in the pictures. The graph of the function $\mathrm{MagicMt}_\pi$ has an interesting self similarity and the nickname of the function $\mathrm{MagicMt}_\pi$ is '**Tsuyama-castle function**' ('津山城関数 **Tsuyama-jyo kansu**' in Japanese). Details of the applications of the function $\mathrm{MagicMt}_\pi$ and the proofs of the ALTEC shall be published elsewhere.

Remarks: The picture of the graph of $\mathrm{MagicMt}_\pi$ in the following Fig 1 and the anaglyph picture of the graph of $\mathrm{MagicMt}_\pi$ given in Fig. 2 were first obtained in the Matrix Art Challenge Seminar in TNCT in parallel with the procedure of what is called the Niagara Project, which is a special new part of the on-going international, interdisciplinary, and inter-generational Second Generation Fukui Project.
In *Fig* 2, we provide Anaglyph Matrix Art of the graph of $\mathrm{MagicMt}_\pi$, which was created in the Matrix Art Challenge Seminar in Tsuyama National College of Technology. The red-blue glasses are needed to see the graph 3-dimentionally. (Cf. ref. [8] and references therein for the origin and background of the Matrix Art, the Niagara Project, and the First Generation and the Second Generation Fukui Project.)

In ref. [8], entitled 'Proof of the Fukui conjecture via resolution of singularities and related methods. V', theory of analytic (highly smooth) curves and resolution of singularities has been applied to prove the Fukui conjecture originating in chemistry. We remark that the investigations of highly smooth functions and of highly irregular functions are complementary in the repeat space theory (RST), which is the central unifying theory in the ongoing Second Generation Fukui Project.

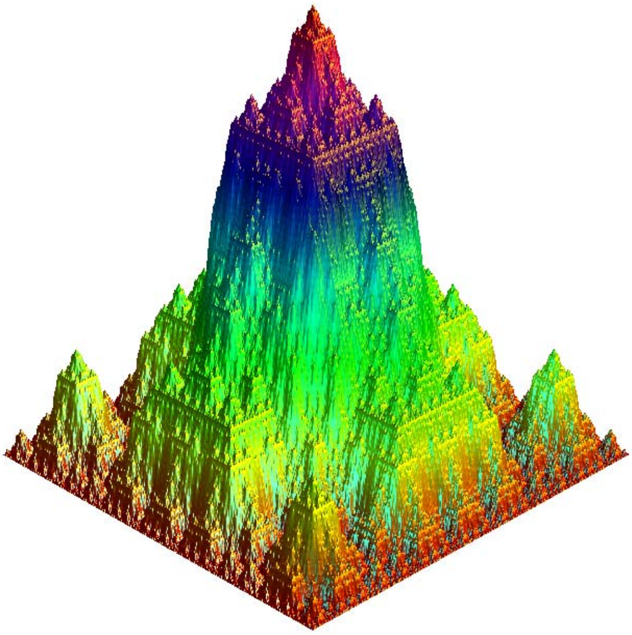

Fig1. Matrix Art picture of the graph of the function $\mathrm{MagicMt}_{\pi}$, nicknamed Tsuyama-castle function

津山城関数

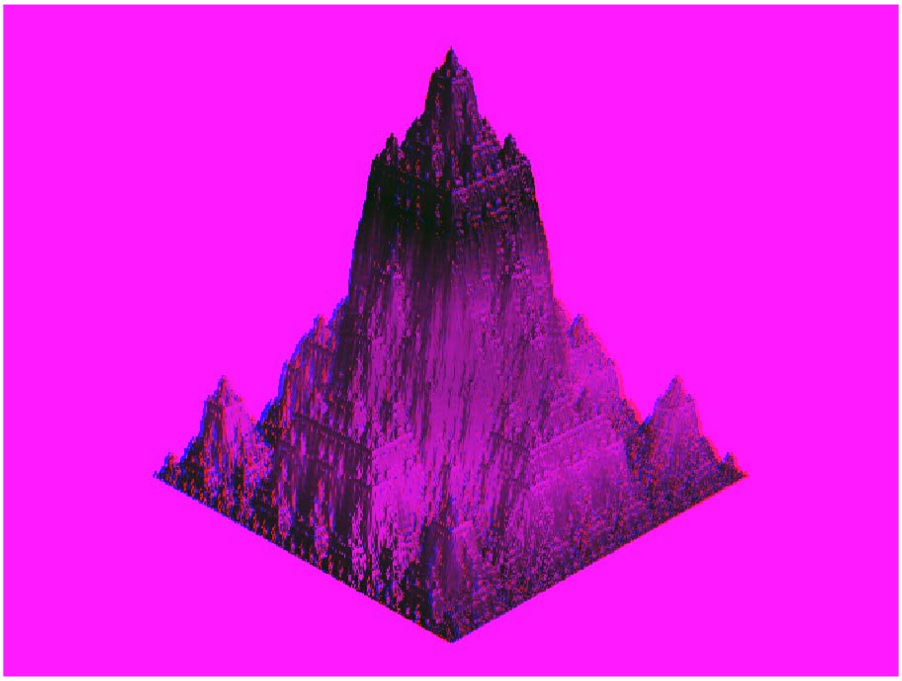

Fig. 2. Anaglyph Matrix Art of the graph of $\mathrm{MagicMt}_{\pi}$

References

[1] S. Arimoto and K. Fukui, Fundamental Mathematical Chemistry, Interdisciplinary Research in Fundamental Mathematical Chemistry and Generalized Repeat Space, IFC Bulletin, Kyoto, 1998, pp 7-13.

[2] S. Arimoto, Note on the repeat space theory - its development and communications with Prof. Kenichi Fukui, J. Math. Chem. 34 (2003) 253-257.

[3] S. Arimoto, Normed repeat space and its super spaces: fundamental notions for the second generation Fukui project, J. Math. Chem. 46 (2009) 589.

[4] S. Arimoto, M. Spivakovsky, K.F. Taylor, and P.G. Mezey, Proof of the Fukui conjecture via resolution of singularities and related methods. I, J. Math. Chem. 37 (2005) 75-91.

[5] S. Arimoto, M. Spivakovsky, K.F. Taylor, and P.G. Mezey, Proof of the Fukui conjecture via resolution of singularities and related methods. II, J. Math. Chem. 37 (2005) 171-189.

[6] S. Arimoto, Proof of the Fukui conjecture via resolution of singularities and related methods. III, J. Math. Chem. 47 (2010) 856.

[7] S. Arimoto, M. Spivakovsky, K.F. Taylor, and P.G. Mezey, Proof of the Fukui conjecture via resolution of singularities and related methods. IV, J. Math. Chem. 48 (2010) 776.

[8] S. Arimoto, M. Spivakovsky, E. Yoshida, K.F. Taylor, and P.G. Mezey, Proof of the Fukui conjecture via resolution of singularities and related methods. V, J. Math. Chem. 49 (2011) 1700.

[9] S. Arimoto, K. Fukui, K.F. Taylor, and P.G. Mezey, Structural Analysis of Certain Linear Operators Representing Chemical Network Systems via the Existence and Uniqueness Theorems of Spectral Resolution. IV, Int. J. Quantum Chem. 67 (1998) 57-69.

[10] S. Arimoto and M. Spivakovsky, The Asymptotic Linearity Theorem for the Study of Additivity Problems of the Zero-point Vibrational Energy of Hydrocarbons and the Total Pi-electron Energy of Alternant Hydrocarbons, J. Math. Chem. 13 (1993) 217-247.

[11] S. Arimoto and K.F Taylor, Aspects of Form and General Topology: Alpha Space Asymptotic Linearity Theorem and the Spectral Symmetry of Alternants, J. Math. Chem. 13 (1993) 249-264.

[12] S. Arimoto and K.F. Taylor, Practical Version of the Asymptotic Linearity Theorem with Applications to the Additivity Problems of Thermodynamic Quantities, J. Math. Chem. 13 (1993) 265-275.

[13] S. Arimoto, New proof of the Fukui conjecture by the Functional Asymptotic Linearity Theorem, J. Math. Chem. 34 (2003) 259.

[14] S. Arimoto, Repeat space theory applied to carbon nanotubes and related molecular networks. I, J. Math. Chem. 41 (2007) 231.

[15] S. Arimoto, Repeat space theory applied to carbon nanotubes and related molecular networks. II, J. Math. Chem. 43 (2008) 658.

[16] S. Arimoto, K. Fukui, P. Zizler, K.F. Taylor and P.G. Mezey, Structural Analysis of Certain Linear Operators Representing Chemical Network Systems via the Existence and Uniqueness Theorems of Spectral Resolution. V, Int. J. Quantum Chem. 74 (1999) 633.

[17] S. Arimoto, M. Spivakovsky, H. Ohno, P. Zizler, K.F. Taylor, T. Yamabe and P.G. Mezey, Structural Analysis of Certain Linear Operators Representing Chemical Network Systems via the Existence and Uniqueness Theorems of Spectral Resolution. VI, Int. J. Quantum Chem. 84 (2001) 389.

[18] S. Arimoto, M. Spivakovsky, H. Ohno, P. Zizler, R.A. Zuidwijk, K.F. Taylor, T. Yamabe and P.G. Mezey, Structural Analysis of Certain Linear Operators Representing Chemical Network Systems via the Existence and Uniqueness Theorems of Spectral Resolution. VII, Int. J. Quantum Chem. 97 (2004) 765.

[19] S. Arimoto, The Functional Delta Existence Theorem and the reduction of a proof of the Fukui conjecture to that of the Special Functional Asymptotic Linearity Theorem, J. Math. Chem. 34 (2003) 287-296.

[20] S. Arimoto, Open problem, Magic Mountain and Devil's Staircase swapping problems, J. Math. Chem. 27 (2000) 213-217.

[21] J.B. Conway, *A course in Functional Analysis* (Springer, New York, 1985).

[22] C. Constantinescu, *C*-Algebras, Vol. 2 and 3* (Elsevier, Amsterdam, 2001).