On an estimate of the Kloosterman zeta function

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In this paper we state an estimate of the Kloosterman zeta function $Z_{m,n}(s,\Gamma)$. Especially we consider the case that Γ is the Hecke congruence group $\Gamma_0(N)$. This case allows us to use the Weil estimate for the Kloosterman sum. From this we can derive an estimate for a certain Dirichlet series consisting of absolute values of Kloosterman sums. Moreover we state two estimates for the K-Bessel function $K_0(y)$. By virtue of these estimates we can obtain an improvement for the order of $|m|$ and $|n|$.

Key words : Kloosterman sum, Kloosterman zeta function

1. Introduction

The Kloosterman zeta function $Z_{m,n}(s,\Gamma)$ plays an important role in the spectral theory of automorphic functions(*cf.* 5), 8), 10)). In particular its evaluation for $|s|$, $|m|$ and $|n|$ in $\Re(s) > 1/2$ is closely related to the problem of estimating "sums of Kloosterman sums". Probably motivated by the works of Bruggeman 1) and Kuznetsov 7), Goldfeld and Sarnak presented a method for evaluating $Z_{m,n}(s,\Gamma)$, and established an estimate(see 2:Theorem 1)). Remarkably their result is applicable to any Fuchsian group Γ of the first kind with a cusp ∞. For this result see also the papers 3:Appendix E) and 14).

In this paper we state the results which hold for the case that Γ is the Hecke congruence group $\Gamma_0(N)$:

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbf{Z}) | c \equiv 0 \pmod N\},$$

and this restriction gives us an improvement for the order of $|m|$ and $|n|$(see Theorem 1.1 and 1.2) when they are compared with those of Goldfeld-Sarnak 2), Hejhal 3) and Yoshida 14).

Let Γ be an arbitrary Fuchsian group of the first kind with a cusp ∞, and let q be the smallest positive number such that the transformation $\left(\begin{smallmatrix} 1 & q \\ & 1 \end{smallmatrix}\right)$ belongs to Γ. For arbitrary nonzero integers m and n, we denote by $S(m,n,c,\Gamma)$ the Kloosterman sum:

$$S(m,n,c,\Gamma) = \sum_{0 \le a,d < qc} e^{2\pi i \frac{ma+nd}{qc}}, \qquad (1.1)$$

where the sum is taken over the elements $\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right) \in \Gamma$ for any fixed $c > 0$. If Γ is the Hecke congruence group $\Gamma_0(N)$, then $q = 1$, and it is well known that the Kloosterman sum (1.1) satisfies the Weil estimate:

$$|S(m,n,c,\Gamma)| \le (|m|,|n|,c)^{1/2} d(c) c^{1/2}, \qquad (1.2)$$

where (, ,) denotes the greatest common divisor, and $d(c)$ the divisor function.

For an arbitrary Fuchsian group Γ of the first kind with a cusp ∞ and for $s \in \mathbf{C}$, Selberg 11) defined the Kloosterman zeta function $Z_{m,n}(s,\Gamma)$ as follows

$$Z_{m,n}(s,\Gamma) = \sum_{c>0} \frac{S(m,n,c,\Gamma)}{c^{2s}}. \qquad (1.3)$$

If $\Gamma = \Gamma_0(N)$, the summand c runs over all positive integers such that $c \equiv 0 \pmod N$, and the Weil estimate (1.2) implies that the series converges absolutely and uniformly for $\Re(s) > 3/4$.

Let $\mathcal{H} = \{z = x+iy \in \mathbf{C} \mid y > 0\}$ be the complex upper half plane given the hyperbolic measure $d\mu(z) = dxdy/y^2$, and $L^2(\Gamma\backslash\mathcal{H})$ the Hilbert space consisting of all functions which are Γ-automorphic and square-integrable for the inner product

$$< f(z), g(z) >= \int_{\Gamma\backslash\mathcal{H}} f(z)\overline{g(z)}\, d\mu(z), \qquad (1.4)$$

where $\Gamma\backslash\mathcal{H}$ is a fundamental domain of Γ and $\overline{g}$ the complex conjugate of g.

The Laplace operator $D := y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$ acts on the space $L^2(\Gamma\backslash\mathcal{H})$ and has spectral decomposition. Then it is known that the eigenvalues $\{\lambda_j\}_{j\ge1}$ under $Df_j = -\lambda_j f_j$ of the discrete spectrum in $L^2(\Gamma\backslash\mathcal{H})$ are real and positive. Here if there exists an eigenvalue such that $0 < \lambda_j < 1/4$, it is called an exceptional eigenvalue. In the paper 11), Selberg has conjectured that if Γ is a congruence subgroup, then the space $L^2(\Gamma\backslash\mathcal{H})$ has no exceptional eigenvalues with respect to the action of D. This conjecture has been proved for $\Gamma = PSL(2,\mathbf{Z})$ by Maass or Roelcke, and by Huxley 4) for Hecke congruence groups $\Gamma_0(N)$ with $N \le 17$(*cf.* 5)).

We write $\lambda_j = s_j(1-s_j)$. If λ_j is exceptional, it gives a real value s_j with $1/2 < s_j < 1$. Let us denote by T_0 the set of such points s_j. Then for

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any positive ε_0 chosen so that $(1/2,(1/2)+2\varepsilon_0)\cap T_0=\emptyset$, we define the domain U_{ε_0} in the complex s-plane to be

$$\{\sigma\,|\,\frac{1}{2}<\sigma<\frac{1}{2}+\varepsilon_0\}\times\{\tau\,|\,|\tau|\le 1\} \tag{1.5}$$

for $s=\sigma+i\tau$.

We show that

Theorem 1.1. *Let Γ be the Hecke congruence group $\Gamma_0(N)$, and m and n be nonzero integers. Moreover put $s=\sigma+i\tau$. Then the Kloosterman zeta function $Z_{m,n}(s,\Gamma)$ is continued meromorphically to the domain $\Re(s)>1/2$ with at most a finite number of simple poles at $s=s_j$ and satisfies the estimate:*

$$Z_{m,n}(s,\Gamma) \tag{1.6}$$
$$=O\Bigg(\frac{|mn|^{1/4}d(|m|N)d(|n|N)}{N^{2\sigma-1/2}} \times|\tau|^{1/2}\frac{1}{(\sigma-\frac{1}{2})^4}\Bigg)$$

for $1/2<\sigma<M$ and $|\tau|\ge 1$, where $d(\cdot)$ denotes the divisor function, M is an arbitrarily fixed positive number such that $M>1/2$ and the implied constant depends only on M, and

$$Z_{m,n}(s,\Gamma) \tag{1.7}$$
$$=O\Bigg(\frac{|mn|^{1/4}d(|m|N)d(|n|N)}{N^{2\sigma-1/2}} \times\frac{1}{(\sigma-\frac{1}{2})^4\sqrt{\tau^2+(\sigma-\frac{1}{2})^2}}\Bigg)$$

for $s\in U_{\varepsilon_0}$, where the implied constant is an absolute constant.

For the case of $T_0=\emptyset$, we may take $\varepsilon_0=M$ in the estimate (1.7), and then the implied constant depends only on M.

Concerning the estimate (1.6) for example, the result of Goldfeld-Sarnak 2:Theorem 1) is $O(|mn||\tau|^{1/2}/(\sigma-1/2))$. Hejhal 3:pp.709) obtained more strict one $O(|m||n|^{1/2}|\tau|^{1/2}/(\sigma-1/2)^2)$, and we proved $O(|mn|^{1/2}|\tau|^{1/2}/(\sigma-1/2)^3)$ in 14:Theorem). As we noted above these results are applicable to any Fuchsian group Γ of the first kind with a cusp ∞. Moreover we shall mention that Goldfeld-Sarnak and Hejhal have considered the same problem and obtained similar results for more general Kloosterman zeta function associated with the Hilbert space $L^2(\Gamma\backslash\mathcal{H},k,\chi)$, $k>0$ and χ being a multiplier for this k.

The following two estimates, plus the basic formulae (2.4) and (2.7), are crucial in the proof of Theorem 1.1. One is the estimates for the K-Bessel function $K_0(y)$ as in Lemma 3.1, and the other is that in (3.14) for a certain Dirichlet series, which has been proved by Iwaniec 6) using the Weil estimate (1.2) as an essential tool. To prove Theorem 1.1 we need to evaluate the norms of the non-holomorphic Poincaré series(see Lemma 3.2) and the term $R_{m,n}(s,s+1/2)$(see (3.2)). Then the norms of the non-holomorphic Poincaré series are evaluated by using these two estimates. So the results of Lemma 3.2 hold only for Hecke congruence groups(or probably for congruence subgroups), while the estimate of the lemma in 14) holds for any Fuchsian group of the first kind with a cusp ∞.

From elementary arguments together with Theorem 1.1 we derive the following estimate for sums of Kloosterman sums:

Theorem 1.2. *Let Γ be the Hecke congruence group $\Gamma_0(N)$ with $N\le 17$. Then since there exists no exceptional eigenvalues, we have*

$$\sum_{c<x}\frac{S(m,n,c,\Gamma)}{c} \tag{1.8}$$
$$=O\left(\frac{|mn|^{1/4}d(|m|N)d(|n|N)}{N^{1/2}}x^{\frac{1}{6}+\varepsilon}\right)$$

for any $\varepsilon>0$, where the implied constant is absolute.

2. Notations and basic Formu-lae

Let Γ be a Fuchsian group of the first kind with a cusp ∞, and $\mathcal{H}$ the complex upper half plane. For $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$ and $z\in\mathcal{H}$, we denote the linear fractional transformation by $\gamma z(:=(az+b)/(cz+d))$, and put $\gamma z=x(\gamma z)+iy(\gamma z)$, i.e., $x(\gamma z)$ or $y(\gamma z)$ is real or imaginary part of $\gamma z\in\mathcal{H}$. Moreover let $\Gamma_\infty:=\{\begin{pmatrix}1&q\ell\\&1\end{pmatrix}|\ell\in\mathbf{Z}\}$ be the stabilizer group of a cusp at infinity.

For $m\in\mathbf{Z}_{\neq 0}$, $z\in\mathcal{H}$ and $s\in\mathbf{C}$, the non-holomorphic Poincaré series is defined by

$$P_m(z,s,\Gamma) \tag{2.1}$$
$$=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}e^{2\pi i\frac{m}{q}x(\gamma z)}e^{-2\pi\frac{|m|}{q}y(\gamma z)}y(\gamma z)^s.$$

This series converges absolutely and uniformly for $\Re(s)>1$ and belongs to the Hilbert space $L^2(\Gamma\backslash\mathcal{H})$. Let $a_m(y,s,n,\Gamma)$ be the nth Fourier coefficient of the series $P_m(z,s,\Gamma)$:

$$a_m(y,s,n,\Gamma)=\frac{1}{q}\int_0^q P_m(x+iy,s,\Gamma)e^{-2\pi i\frac{n}{q}x}\,dx.$$

Here we know that the $a_m(\cdot,y,\cdot)$ is expressed as

$$a_m(y,s,n,\Gamma)=\delta_{m,n}y^s e^{-2\pi\frac{|n|}{q}y} \tag{2.2}$$

$$+\frac{1}{q}\sum_{c>0}\frac{S(m,n,c,\Gamma)}{c^{2s}}y^{1-s}$$
$$\times\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^s}$$
$$\times e\left(\frac{|m|}{qc^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)e^{2\pi i\frac{n}{q}xy}\,dx,$$

where $\delta_{m,n}$ is the Kronecker symbol, ϵ_1 indicates 1 or -1 according to $m>0$ and $m<0$, and $e(x)=e^{2\pi ix}$.

As is developed in Goldfeld-Sarnak 2), we consider the inner product of the non-holomorphic Poincaré series: $< P_m(z,s,\Gamma), P_n(z,\overline{w},\Gamma) >$ for $m,\ n\in\mathbf{Z}_{\neq 0}$ and $s,\ w\in\mathbf{C}$(see (1.4)). Then the Rankin-Selberg method yields that

$$< P_m(z,s,\Gamma), P_n(z,\overline{w},\Gamma) > \tag{2.3}$$
$$= q\int_0^{\infty} a_m(y,s,n,\Gamma)y^{w-2}e^{-2\pi\frac{|n|}{q}y}\,dy$$

for $\Re(s)>1$ and $\Re(w)>1$. Substituting the expression (2.2) into the integrand we obtain

$$< P_m(z,s,\Gamma), P_n(z,\overline{w},\Gamma) > \tag{2.4}$$
$$= q\delta_{m,n}\left(4\pi\frac{|m|}{q}\right)^{1-s-w}\Gamma(s+w-1)$$
$$+\sum_{c>0}\frac{S(m,n,c,\Gamma)}{c^{2s}}\int_0^{\infty}y^{w-s-1}e^{-2\pi\frac{|n|}{q}y}\,dy$$
$$\times\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^s}$$
$$\times e\left(\frac{|m|}{qc^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)e^{2\pi i\frac{n}{q}xy}\,dx$$

for $\Re(s)>1$ and $\Re(w)>1$.

We return to the expression (2.2). Following the idea of Goldfeld-Sarnak 2) we decompose the second term into two terms:

$$\frac{1}{q}\sum_{c>0}\frac{S(m,n,c,\Gamma)}{c^{2s}}y^{1-s} \tag{2.5}$$
$$\times\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^s}e^{2\pi i\frac{n}{q}xy}\,dx$$
$$+\frac{1}{q}\sum_{c>0}\frac{S(m,n,c,\Gamma)}{c^{2s}}y^{1-s}\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^s}$$
$$\times\left(e\left(\frac{|m|}{qc^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\right)e^{2\pi i\frac{n}{q}xy}\,dx.$$

Let K_ν be the K-Bessel function. One knows from 6:pp.60, (3.19)) that

$$K_\nu(|y|) = 2^{\nu-1}\pi^{-\frac{1}{2}}|y|^{-\nu}\Gamma(\nu+\frac{1}{2}) \tag{2.6}$$
$$\times\int_{-\infty}^{\infty}\frac{e^{iyx}}{(1+x^2)^{\nu+\frac{1}{2}}}\,dx$$

for $\Re(\nu)>0$ and $y\in\mathbf{R}_{\neq\mathbf{0}}$. Using this we see that the first term in the above is equal to

$$\frac{1}{q}2\pi^{\frac{1}{2}}\left(\pi\frac{|n|}{q}\right)^{s-\frac{1}{2}}Z_{m,n}(s,\Gamma)$$
$$\times\frac{1}{\Gamma(s)}y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi\frac{|n|}{q}y).$$

Moreover we know the formula

$$\int_0^{\infty}K_\nu(y)e^{-y}y^{s-1}\,dy = \pi^{\frac{1}{2}}2^{-s}\frac{\Gamma(s+\nu)\Gamma(s-\nu)}{\Gamma(s+\frac{1}{2})}.$$

Therefore if we substitute the expression (2.5) into (2.3), we have

$$< P_m(z,s,\Gamma), P_n(z,\overline{w},\Gamma) > \tag{2.7}$$
$$= q\delta_{m,n}\left(4\pi\frac{|m|}{q}\right)^{1-s-w}\Gamma(s+w-1)$$
$$+2^{2-2w}\pi\left(\pi\frac{|n|}{q}\right)^{s-w}$$
$$\times\frac{\Gamma(s+w-1)\Gamma(w-s)}{\Gamma(s)\Gamma(w)}Z_{m,n}(s,\Gamma)$$
$$+R_{m,n}(s,w)$$

for $\Re(s)>1$ and $\Re(w)>1$, where

$$R_{m,n}(s,w) := \sum_{c>0}\frac{S(m,n,c,\Gamma)}{c^{2s}} \tag{2.8}$$
$$\times\int_0^{\infty}y^{w-s-1}e^{-2\pi\frac{|n|}{q}y}\,dy$$
$$\times\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^s}$$
$$\times\left(e\left(\frac{|m|}{qc^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\right)e^{2\pi i\frac{n}{q}xy}\,dx.$$

Formulas (2.4) and (2.7) with (2.8) are the main tools in our proof of Theorem 1.1.

The non-holomorphic Poincaré series P_m defined by (2.1) satisfies the recursion relation

$$\{s(1-s)+D\}P_m(z,s,\Gamma)$$
$$= -4\pi\frac{|m|}{q}sP_m(z,s+1,\Gamma)$$

for $\Re(s)>1$. This is equivalent to

$$P_m(z,s,\Gamma) = -4\pi\frac{|m|}{q}s\mathcal{R}_\lambda P_m(z,s+1,\Gamma), \tag{2.9}$$

where $\mathcal{R}_\lambda := (\lambda+D)^{-1}(\lambda=s(1-s))$ is the resolvent of D. It is known that $\mathcal{R}_\lambda$ is holomorphic for $\Re(s)>1/2$ except for simple poles at $s=s_j$ and satisfies the estimates:

$$\|\ \mathcal{R}_\lambda\ \|\leq\frac{1}{|\tau|(2\sigma-1)} \tag{2.10}$$

for $\sigma > 1/2$ and $|\tau| \geq 1$, and

$$\| \mathcal{R}_\lambda \| \leq \frac{1}{(\sigma - \frac{1}{2})\sqrt{\tau^2 + (\sigma - \frac{1}{2})^2}} \tag{2.11}$$

for $s \in U_{\varepsilon_0}$, where U_{ε_0} is the set defined by (1.5). Concerning this property the reader is referred to 2:pp.248)(see also 3:pp.672), 9), 10)).

3. Proof of Theorem 1.1 and 1.2

Throughout this section we always assume that Γ is the Hecke congruence group $\Gamma_0(N)$. Thus $q = 1$ in the formula (2.7). Moreover putting $w = s+1/2$ we have

$$\begin{aligned} &2^{1-2s}\pi(\pi|n|)^{-\frac{1}{2}} \\ &\quad \times \frac{\Gamma(2s-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(s)\Gamma(s+\frac{1}{2})} Z_{m,n}(s,\Gamma) \\ &\quad = -\delta_{m,n}(4\pi|m|)^{\frac{1}{2}-2s}\Gamma(2s - \frac{1}{2}) \\ &\qquad + < P_m(z,s,\Gamma), P_n(z,\overline{s}+\frac{1}{2},\Gamma > \\ &\qquad -R_{m,n}(s,s+\frac{1}{2}) \end{aligned} \tag{3.1}$$

for $\Re(s) > 1$, where

$$\begin{aligned} &R_{m,n}(s,s+\frac{1}{2}) \\ &= \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{S(m,n,c,\Gamma)}{c^{2s}} \\ &\quad \times \int_0^\infty y^{\frac{1}{2}-1} e^{-2\pi|n|y}\, dy \\ &\quad \times \int_{-\infty}^\infty \frac{1}{(1+x^2)^s} \\ &\quad \times \left(e\left(\frac{|m|}{c^2} \cdot \frac{\epsilon_1 x + i}{y(x^2+1)}\right) - 1 \right) e^{2\pi i n x y}\, dx. \end{aligned} \tag{3.2}$$

It follows from (2.9) that

$$\begin{aligned} < P_m(z,s,\Gamma), P_n(z,\overline{s}+\frac{1}{2},\Gamma) >&= -4\pi|m|s \\ \times < \mathcal{R}_\lambda P_m(z,s+1,\Gamma), &P_n(z,\overline{s}+\frac{1}{2},\Gamma) >, \end{aligned}$$

where $\lambda = s(1-s)$. Thus the left-hand side is continued meromorphically to the domain $\Re(s) > 1/2$ by the right-hand side since $\mathcal{R}_\lambda$ is meromorphic there. Moreover by the Cauchy-Schwarz inequality we have

$$\begin{aligned} &| < P_m(z,s,\Gamma), P_n(z,\overline{s}+\frac{1}{2},\Gamma) > | \\ &\quad \leq 4\pi|m||s| \cdot \| \mathcal{R}_\lambda \| \\ &\quad \times \| P_m(z,s+1,\Gamma) \| \cdot \| P_n(z,\overline{s}+\frac{1}{2},\Gamma) \| \end{aligned} \tag{3.3}$$

for $\Re(s) > 1/2$. We need to evaluate the norms of P_m and P_n. To do this we prepare the following lemma.

Recall that the the K-Bessel function is defined by

$$K_\nu(y) = \frac{1}{2}\left(\frac{y}{2}\right)^\nu \int_0^\infty e^{-t} e^{-\frac{y^2}{4t}} t^{-\nu-1}\, dt \tag{3.4}$$

for any $\nu \in \mathbf{C}$ and $y > 0$. Then we have

Lemma 3.1. *Let $y > 0$ be a positive number and $\nu = 0$ in (3.4). Then the K-Bessel function $K_0(y)$ satisfies the following two estimates:*

$$K_0(y) \ll y^{-3/2} \tag{3.5}$$

and

$$K_0(y) \ll y^{-1/2} \tag{3.6}$$

for any $y > 0$, where the implied constants are absolute.

Proof. We first prove (3.5). To the formula (2.6) for K_ν if we apply partial integration, it is transformed into

$$\begin{aligned} K_\nu(y) = -2^\nu \pi^{-\frac{1}{2}} i y^{-\nu-1}\Gamma(\nu+\frac{3}{2}) \\ \times \int_{-\infty}^\infty \frac{x e^{iyx}}{(1+x^2)^{\nu+\frac{3}{2}}}\, dx \end{aligned}$$

for $\Re(\nu) > -1/2$ and $y > 0$. Repeating this transform we obtain

$$\begin{aligned} K_\nu(y) = O\bigg(&|\Gamma(\nu+\frac{1}{2}+k)| \\ &\times y^{-\Re(\nu)-k}(1+\frac{1}{\Re(\nu)+\frac{k}{2}}) \bigg) \end{aligned} \tag{3.7}$$

for $\Re(\nu) > -k/2$ and $y > 0$, where the implied constant is absolute, and the integer k is $k \geq 0$ if $\Re(\nu) > 0$ and $k \geq 1$ if $\Re(\nu) = 0$. Choosing $k = 1$ we have $K_0(y) \ll y^{-1}$. However it is clear that $y^{-1} \leq y^{-3/2}$ if $y \leq 1$. So we have $K_0(y) \ll y^{-3/2}$ for $y \leq 1$. If $y \geq 1$, we first have $K_0(y) \ll y^{-2}$ by choosing $k = 2$ in (3.7). But $y^{-2} \leq y^{-3/2}$ since $y \geq 1$. Hence we obtain $K_0(y) \ll y^{-3/2}$ for any $y > 0$. Next we prove another estimate (3.6). One knows the integral representation:

$$\begin{aligned} K_{\nu-\mu}(y) = 2^\mu \Gamma(\mu+1) y^{\mu-\nu} \\ \times \int_0^\infty x^{\nu+1} \frac{1}{(x^2+y^2)^{\mu+1}} J_\nu(x)\, dx \end{aligned}$$

for $-1 < \Re(\nu) < 2\Re(\mu) + 1/2$(see 12:pp.434,(2))), where J_ν is the Bessel function defined for example by

$$\begin{aligned} J_\nu(x) = \pi^{-\frac{1}{2}} \frac{1}{\Gamma(\nu+\frac{1}{2})}(\frac{x}{2})^\nu \\ \times \int_{-1}^1 e^{ixt}(1-t^2)^{\nu-\frac{1}{2}}\, dt \end{aligned} \tag{3.8}$$

for $\Re(\nu) > -1/2$ and $x > 0$. Put $\nu = \mu = 1/4$. Noticing $(x^2+y^2)^{-(1/4+1)} \leq x^{-2}y^{-1/2}$ we have

$$K_0(y) \ll y^{-\frac{1}{2}} \int_0^\infty x^{\frac{1}{4}-1} |J_{\frac{1}{4}}(x)|\, dx.$$

For the Bessel function $J_{1/4}$ we derive from (3.8) that $J_{1/4}(x) \ll x^{1/4}$, and moreover it is well known that $J_\nu(x) \ll x^{-1/2}$ as x tends to ∞. Thus the integral over x is convergent with an absolute constant. Therefore we obtain $K_0(y) \ll y^{-1/2}$ for any $y>0$. □

For the norms of P_m and P_n we state the following

LEMMA 3.2. *Let M be a positive constant such that $M>1/2$. Then we have*

$$\| P_m(z, s+1, \Gamma) \| = O\left(\frac{|m|^{-3/4} d(|m|N)^{1/2}}{N^{\sigma-1/4}} \sqrt{1 + \frac{1}{(\sigma-\frac{1}{2})^2}} \right) \tag{3.9}$$

for $1/2<\sigma<M$, and

$$\| P_n(z, \overline{s}+\frac{1}{2}, \Gamma) \| = O\left(\frac{|n|^{-1/4} d(|n|N)^{1/2}}{N^{\sigma-1/4}} \sqrt{1 + \frac{1}{(\sigma-\frac{1}{2})^2}} \right) \tag{3.10}$$

for $1/2<\sigma<M$, where the implied constants depend only on M.

PROOF. Since $\| P_m(z,s+1,\Gamma) \|^2 = < P_m(z,s+1,\Gamma), P_m(z,s+1,\Gamma) >$, the formula (2.4) yields that

$$\begin{aligned}\| P_m(z, s+1, \Gamma) \|^2 &= (4\pi|m|)^{-2\sigma-1}\Gamma(2\sigma+1) \\ &+ \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{S(m,m,c,\Gamma)}{c^{2(s+1)}} \\ &\times \int_0^\infty y^{-2i\tau-1} e^{-2\pi|m|y}\, dy \\ &\times \int_{-\infty}^\infty \frac{1}{(1+x^2)^{(s+1)}} \\ &\times e\left(\frac{|m|}{c^2} \cdot \frac{\epsilon_1 x + i}{y(x^2+1)}\right) e^{2\pi i m x y}\, dx\end{aligned}$$

for $s = \sigma + i\tau$. Then the absolute value of the multiple integral over y and x is majorized by

$$\begin{aligned}&\int_0^\infty y^{-1} e^{-2\pi|m|y}\, dy \\ &\times \int_{-\infty}^\infty \frac{1}{(1+x^2)^{(\sigma+1)}} \\ &\times \exp\left(-2\pi\frac{|m|}{c^2(x^2+1)}\frac{1}{y}\right)\, dx.\end{aligned} \tag{3.11}$$

Here from (3.4) we see that the integral over y is equal to $2K_0(4\pi\frac{|m|}{c}\frac{1}{\sqrt{x^2+1}})$. Thus we derive the inequality:

$$\begin{aligned}&\| P_m(z, s+1, \Gamma) \|^2 \\ &\ll |m|^{-2\sigma-1}\Gamma(2\sigma+1) \\ &+ \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{|S(m,m,c,\Gamma)|}{c^{2\sigma+2}} \\ &\times \int_{-\infty}^\infty \frac{1}{(1+x^2)^{(\sigma+1)}} K_0(4\pi\frac{|m|}{c}\frac{1}{\sqrt{x^2+1}})\, dx.\end{aligned} \tag{3.12}$$

We apply here the estimate (3.5) to (3.12). Thus we have

$$\begin{aligned}&\| P_m(z, s+1, \Gamma) \|^2 \\ &\ll |m|^{-2\sigma-1}\Gamma(2\sigma+1) \\ &+|m|^{-3/2} \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{|S(m,m,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}} \\ &\times \int_{-\infty}^\infty \frac{1}{(1+x^2)^{(\sigma+\frac{1}{4})}}\, dx.\end{aligned} \tag{3.13}$$

For the sum over c, Iwaniec 6:pp.169) has established the following by using the Weil estimate (1.2):

$$\begin{aligned}&\sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}} \\ &\ll \min(d(|m|N), d(|n|N)) \\ &\times N^{\frac{1}{2}-2\sigma} \left(1 + \frac{1}{(\sigma-\frac{1}{2})^2}\right)\end{aligned} \tag{3.14}$$

for $\sigma>1/2$, where $d(\cdot)$ is the divisor function and the implied constant is absolute. Combining this estimate with (3.13) we conclude (3.9). We turn to the estimate (3.10). From (2.4) we have

$$\begin{aligned}\| P_n(z, \overline{s}+\frac{1}{2}, \Gamma) \|^2 &= (4\pi|n|)^{-2\sigma}\Gamma(2\sigma) \\ &+ \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{S(n,n,c,\Gamma)}{c^{2\overline{s}+1}} \\ &\times \int_0^\infty y^{2i\tau-1} e^{-2\pi|n|y}\, dy \\ &\times \int_{-\infty}^\infty \frac{1}{(1+x^2)^{(\overline{s}+\frac{1}{2})}} \\ &\times e\left(\frac{|n|}{c^2} \cdot \frac{\epsilon_1 x + i}{y(x^2+1)}\right) e^{2\pi i n x y}\, dx\end{aligned}$$

for $s = \sigma + i\tau$. By similar process to (3.11) and (3.12) we obtain

$$\begin{aligned}\| P_n(z, \overline{s}+\frac{1}{2}, \Gamma) \|^2 &\ll |n|^{-2\sigma}\Gamma(2\sigma) \\ &+ \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} \frac{|S(n,n,c,\Gamma)|}{c^{2\sigma+1}}\end{aligned} \tag{3.15}$$

$$\times \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{(\sigma+\frac{1}{2})}} K_0(4\pi\frac{|n|}{c}\frac{1}{\sqrt{x^2+1}})\,dx.$$

Applying the estimate (3.6) to (3.15) we have

$$\| P_n(z,\overline{s}+\frac{1}{2},\Gamma) \|^2 \ll |n|^{-2\sigma}\Gamma(2\sigma) + |n|^{-1/2} \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{|S(n,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}} \times \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{(\sigma+\frac{1}{4})}}\,dx.$$

Hence from using the estimate (3.14) we deduce (3.10). □

From this lemma and (3.3) we derive that

$$| < P_m(z,s,\Gamma), P_n(z,\overline{s}+\frac{1}{2},\Gamma) > | \ll \frac{|m|^{1/4}|n|^{-1/4}d(|m|N)^{1/2}d(|n|N)^{1/2}}{N^{2\sigma-1/2}} \times |s| \cdot \| \mathcal{R}_\lambda \| \left(1+\frac{1}{(\sigma-\frac{1}{2})^2}\right) \tag{3.16}$$

for $\Re(s)=\sigma>1/2$.

We next consider the quantity $R_{m,n}$ in (3.2). Making a change of variable $|m|^{1/2}|n|^{-1/2}y$ for y we have

$$R_{m,n}(s,s+\frac{1}{2}) = |m|^{1/4}|n|^{-1/4} \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{S(m,n,c,\Gamma)}{c^{2s}} \times \int_0^{\infty} y^{\frac{1}{2}-1}e^{-2\pi|mn|^{1/2}y}\,dy \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^s} \times \left(e\left(\frac{|mn|^{1/2}}{c^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\right) \times e^{2\pi in|m|^{1/2}|n|^{-1/2}xy}\,dx. \tag{3.17}$$

We decompose the path of integration over y as follows

$$\begin{aligned} &|m|^{1/4}|n|^{-1/4} \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{S(m,n,c,\Gamma)}{c^{2s}} \times \int_0^{1/c} y^{\frac{1}{2}-1}e^{-2\pi|mn|^{1/2}y}\,dy \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^s} \\ &\times \left(e\left(\frac{|mn|^{1/2}}{c^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\right) \times e^{2\pi in|m|^{1/2}|n|^{-1/2}xy}\,dx \\ &+|m|^{1/4}|n|^{-1/4} \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{S(m,n,c,\Gamma)}{c^{2s}} \times \int_{1/c}^{\infty} y^{\frac{1}{2}-1}e^{-2\pi|mn|^{1/2}y}\,dy \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^s} \\ &\times \left(e\left(\frac{|mn|^{1/2}}{c^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\right) \times e^{2\pi in|m|^{1/2}|n|^{-1/2}xy}\,dx. \end{aligned} \tag{3.18}$$

Since

$$e\left(\frac{|mn|^{1/2}}{c^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1 \ll 1, \tag{3.19}$$

the first term is majorized by

$$\begin{aligned} &|m|^{1/4}|n|^{-1/4} \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma}} \times \int_0^{1/c} y^{\frac{1}{2}-1}\,dy \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\sigma}dx \\ &\ll |m|^{1/4}|n|^{-1/4} \times \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}} \left(1+\frac{1}{(\sigma-\frac{1}{2})}\right) \end{aligned}$$

for $\Re(s)=\sigma>1/2$. Thus by using the estimate (3.14) we conclude that the first term in (3.18) has the estimate:

$$O\Bigg(|m|^{1/4}|n|^{-1/4} \times \min(d(|m|N), d(|n|N)) \times N^{\frac{1}{2}-2\sigma}\left(1+\frac{1}{(\sigma-\frac{1}{2})^3}\right)\Bigg) \tag{3.20}$$

for $\sigma>1/2$. For the second term in (3.18) we divide the sum over c into $c\le|mn|^{1/2}$ and $c>|mn|^{1/2}$. Applying the estimate (3.19) again we see that the sum over $c\le|mn|^{1/2}$ is majorized by

$$|m|^{1/4}|n|^{-1/4} \times \sum_{c\equiv 0\,(\mathrm{mod}\ N), c\le|mn|^{1/2}} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma}} \times \int_{1/c}^{\infty} y^{\frac{1}{2}-1}e^{-2\pi|mn|^{1/2}y}\,dy \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^\sigma}\,dx. \tag{3.21}$$

Here we have by partial integration

$$\int_{1/c}^{\infty} y^{\frac{1}{2}-1}e^{-2\pi|mn|^{1/2}y}\,dy \ll |mn|^{-1/2}c^{1/2}.$$

Thus the quantity in (3.21) is majorized by

$$\begin{aligned} &|m|^{1/4}|n|^{-1/4}|mn|^{-1/2} \times \sum_{c\equiv 0\,(\mathrm{mod}\ N), c\le|mn|^{1/2}} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}}c \times \left(1+\frac{1}{(\sigma-\frac{1}{2})}\right) \\ &\le |m|^{1/4}|n|^{-1/4} \times \sum_{c\equiv 0\,(\mathrm{mod}\ N)} \frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}}\left(1+\frac{1}{(\sigma-\frac{1}{2})}\right). \end{aligned}$$

Therefore we obtain the same estimate as that of (3.20) by using (3.14). We turn to the sum over

$c>|mn|^{1/2}$. Since $y>1/c$ and $c>|mn|^{1/2}$ we have

$$e\left(\frac{|mn|^{1/2}}{c^2}\cdot\frac{\epsilon_1 x+i}{y(x^2+1)}\right)-1\ll\frac{|mn|^{1/2}}{c^2}\frac{1}{y}\frac{|x|+1}{(x^2+1)}.$$

Thus the term in question is majorized by

$$\begin{aligned}&|m|^{1/4}|n|^{-1/4}|mn|^{1/2}\\&\times\sum_{c\equiv 0\,(\mathrm{mod}\ N),c>|mn|^{1/2}}\frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+2}}\\&\times\int_{1/c}^{\infty}y^{-\frac{1}{2}-1}\,dy\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{\sigma}}\frac{|x|+1}{(x^2+1)}\,dx\\&\le|m|^{1/4}|n|^{-1/4}|mn|^{1/2}\\&\times\sum_{c\equiv 0\,(\mathrm{mod}\ N),c>|mn|^{1/2}}\frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{3}{2}}}\\&\times\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{\sigma}}\frac{|x|+1}{(x^2+1)}\,dx\\&\ll|m|^{1/4}|n|^{-1/4}\\&\times\sum_{c\equiv 0\,(\mathrm{mod}\ N)}\frac{|S(m,n,c,\Gamma)|}{c^{2\sigma+\frac{1}{2}}}\end{aligned}$$

for $\sigma>1/2$. Hence we obtain the estimate which has $(\sigma-1/2)^2$ instead of $(\sigma-1/2)^3$ in (3.20). From these arguments we conclude the fact that the function $R_{m,n}(s,s+1/2)$ defined by (3.2) or (3.17) is holomorphic for $\Re(s)>1/2$ and satisfies the estimate (3.20).

We are ready to prove Theorem 1.1. In (3.1), Stirling's formula gives $\Gamma(2s-1/2)/\Gamma(s)\Gamma(s+1/2)\sim|\tau|^{-1/2}$ and $\Gamma(2s-1/2)\sim|\tau|^{2\sigma-1}e^{-\pi|\tau|}$ for $s=\sigma+i\tau$ and $|\tau|\ge 1$. Hence the estimates (3.16),(3.20),(2.10) and (2.11) deduce our assertion.

We next prove Theorem 1.2. Let x be a sufficiently large positive number, and T a positive number satisfying $1\le T\le x^{1/2}$. Then the Weil estimate (1.2) and the arguments similar to those of Hejhal 3:pp.697-699) yield that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon_0-iT}^{\frac{1}{2}+\varepsilon_0+iT}Z_{m,n}\Big(\frac{1+s}{2},\Gamma\Big)\frac{x^s}{s}\,ds \qquad (3.22)\\&=\sum_{c<x}\frac{S(m,n,c,\Gamma)}{c}\\&+O\left(\frac{(|m|,|n|)^{1/2}}{N}\cdot\frac{x^{\frac{1}{2}}\log x}{T}\right),\end{aligned}$$

where $\varepsilon_0=(\log x)^{-1}$. Let E be the rectangle defined by $E=[\varepsilon_0,\frac{1}{2}+\varepsilon_0]\times[-T,T]$. Since $N\le 17$ the space $L^2(\Gamma_0(N)\backslash\mathcal{H})$ has no exceptional eigenvalues, and this implies that the Kloosterman zeta function $Z_{m,n}$ is holomorphic in E. Thus we have

$$\frac{1}{2\pi i}\int_{\partial E}Z_{m,n}\Big(\frac{1+s}{2},\Gamma\Big)\frac{x^s}{s}\,ds=0.$$

Here the Phragmén-Lindelöf principle based on Theorem 1.1 and (3.14) gives

$$\begin{aligned}&\left|Z_{m,n}\Big(\frac{1+s}{2},\Gamma\Big)\right|\\&=O\Bigg(\frac{|mn|^{1/4}d(|m|N)d(|n|N)N^{-1/2}}{\varepsilon_0^4}\\&\qquad\times|\tau|^{\frac{1}{2}+\varepsilon_0-\sigma}\Bigg)\end{aligned}$$

for $\varepsilon_0\le\sigma\le\frac{1}{2}+\varepsilon_0$. By using this estimate and (1.7) we derive that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac{1}{2}+\varepsilon_0-iT}^{\frac{1}{2}+\varepsilon_0+iT}Z_{m,n}\Big(\frac{1+s}{2},\Gamma\Big)\frac{x^s}{s}\,ds \qquad (3.23)\\&\ll\frac{|mn|^{1/4}d(|m|N)d(|n|N)}{N^{1/2}}\\&\times\left(\frac{x^{\frac{1}{2}+\varepsilon}}{T}+T^{1/2}x^{\varepsilon}\right)\end{aligned}$$

for $\forall\varepsilon>0$. Since $|mn|^{1/4}d(|m|N)d(|n|N)\ge(|m|,|n|)^{1/2}$, the formulas (3.22) and (3.23) deduce Theorem 1.2 by taking $T=x^{1/3}$.

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