

The Second Generation Fukui Project and a New Application of the Asymptotic Linearity Theorem[†]

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The Asymptotic Linearity Theorem (ALT), which proves the Fukui conjecture in a broader context, plays a significant role in the repeat space theory (RST) - the central unifying theory in the first and the second generation Fukui Project. The present paper provides a new application of the ALT. This application is important for a new development of the RST towards the solution of Fukui's DNA problem. The present paper also reviews the notions of the normed repeat space $\mathcal{A}(q, d, p)$ and its super space $\mathcal{B}(q, d, p)$, which are special Banach algebras fundamental in the second generation Fukui project.

KEYWORDS: the Fukui conjecture, repeat space theory (RST), Asymptotic Linearity Theorem (ALT), Banach algebras

1. INTRODUCTION

In his later years, Kenichi Fukui (1918 - 1998, Nobel Prize 1981) presented several conjectures concerning the additivity problems of molecules having many identical moieties. Among them is the following which has been playing a significant role in the development of the repeat space theory (RST) (cf. [1-18]), which is the central unifying theory in the first (cf. [1,2]) and the second (cf. [2,3]) generation Fukui Project:

The Fukui Conjecture. *Let $\{M_N\}$ be a fixed element of the repeat space with block-size q , and let I be a fixed closed interval on the real line such that I contains all the eigenvalues of M_N for all positive integers N . Let $\phi_{1/2}: I \rightarrow \mathbb{R}$ denote the function defined by $\phi_{1/2}(t) = |t|^{1/2}$. Then, there exist real numbers α and β such that*

$$\mathrm{Tr} \phi_{1/2}(M_N) = \alpha N + \beta + o(1) \quad (1.1)$$

as $N \rightarrow \infty$.

Fukui's DNA problem, which is closely related to the Fukui conjecture above, is a long-range target of the first and second generation Fukui Project, whose underlying motive has been to cultivate a new interdisciplinary region between chemistry and mathematics, especially for tackling what we call *globally-pertaining-type problems*, or, for short, *g-type problems* [2]; these constitute physicochemical problems for whose solutions global mathematical contextualization is essential. "Can the conductivity and

other properties of a single-walled carbon nanotube be analyzed in the setting of a *-algebra equipped with a complete metric?" This metric problem is fundamental to proceed towards the solution of Fukui's DNA problem. In recent publication [3] by the present author, this metric problem was affirmatively solved and the new notion of normed repeat space $\mathcal{A}(q, d, p)$ was established. The normed repeat space $\mathcal{A}(q, d, p)$ is an intermediate theoretical device to shift from periodic polymers to aperiodic polymers like DNA and RNA in the Fukui Project. The space $\mathcal{A}(q, d, p)$ is a Banach algebra for all $1 \leq p \leq \infty$, and $\mathcal{A}(q, d, p)$ forms a C*-algebra for $p = 2$. Here, polymer moiety size number q and dimension number d are arbitrarily given positive integers. The generalized repeat space $\mathcal{A}(q, d)$ is contained in the normed repeat space $\mathcal{A}(q, d, p)$, which in turn is contained in one of its super spaces $\mathcal{B}(q, d, p)$ so that aperiodic polymers can be represented and investigated within this super space $\mathcal{B}(q, d, p)$.

The normed repeat space $\mathcal{A}(q, d, p)$ and its super space $\mathcal{B}(q, d, p)$ are fundamental in the second generation Fukui project.

Let q be any positive integer, the repeat space with block-size q , given in the Fukui conjecture stated above, is denoted by $X(q)$. Let $X(q)$ denote the set of all matrix sequences whose N -th term is an arbitrary $qN \times qN$ real symmetric matrix. Then, one can easily verify that the repeat space $X(q)$ with block-size q is expressed by $X(q) = X(q) \cap \mathcal{A}(q, 1)$. Thus, we have the following relations between the repeat space $X(q)$ in the Fukui conjecture, generalized repeat space $\mathcal{A}(q, 1)$, normed repeat space $\mathcal{A}(q, 1, p)$ and its superspace $\mathcal{B}(q, 1, p)$ (cf. the appendix for the definitions of the latter three spaces):

$$X(q) \subset \mathcal{A}(q, 1) \subset \mathcal{A}(q, 1, p) := \text{closure of } \mathcal{A}(q, 1) \subset \mathcal{B}(q, 1, p). \quad (1.2)$$

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The Asymptotic Linearity Theorem (ALT) plays a significant role in the repeat space theory. This theorem, which was proved by the present author for the first time (cf. [12] and references therein), implies the validity of the Fukui conjecture; combined with its associated theorems, it solves a variety of molecular network problems in a unifying manner (cf. [9,12] and references therein).

We retain the notation of [12]. (The reader is asked to briefly review [12] for the definition of symbols.) The present paper provides a new application of the ALT. The following theorem 1, from which fundamental theorem I in [1] easily follows, is important in a new development of the repeat space theory especially towards the solution of Fukui's DNA problem. The goal of the present paper is to give an affirmative answer to the following problem.

Problem 1. Is it possible to derive theorem 1 by using the ALT?

Theorem 1. Let $a, b \in \mathbb{R}$ with $a < b$, let $x(N, r) := a + (b - a)r/N$, and let $f \in AC[a, b]$. Then, we have

$$\sum_{r=1}^N f(x(N, r)) = (1/(b - a)) \left(\int_a^b f(\theta) d\theta \right) N + (1/2)(f(b) - f(a)) + o(1) \quad (1.3)$$

as $N \rightarrow \infty$.

In section 2, we prepare some tools for the affirmative solution to the above problem. The solution is given in section 3 by deriving, from the ALT, the following theorem 2, from which theorem 1 easily follows.

Theorem 2. Let $a, b \in \mathbb{R}$ with $a < b$, let $x(N, r) := a + (b - a)r/N$, and let g let $f \in AC[a, b]$. Then, there exist real numbers $\alpha(f)$ and $\beta(f)$ such that

$$\sum_{r=1}^N f(x(N, r)) = \alpha(f)N + \beta(f) + o(1) \quad (1.4)$$

as $N \rightarrow \infty$.

2. PREPARATION FOR A SOLUTION OF PROBLEM 1

Throughout, let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, denote respectively the set of all positive integers, nonnegative integers, integers, real numbers, and complex numbers. Let us first recall the symbols we need in this section: Let $a, b \in \mathbb{R}$ with $a < b$, let $I = [a, b]$. The symbol $C(I)$ denotes the set of all real-valued continuous functions on I . A function $\varphi: I \rightarrow \mathbb{R}$ is said to be absolutely continuous on I if, given any $\varepsilon > 0$, there

exists a $\delta > 0$ such that for every finite system of pairwise disjoint subintervals $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \subset [a, b]$,

$$\sum_{k=1}^n (b_k - a_k) < \delta$$

implies

$$\sum_{k=1}^n |\varphi(b_k) - \varphi(a_k)| < \varepsilon.$$

The symbol $AC(I)$ denotes the set of all real-valued absolutely continuous functions on I . If f is a real-valued function on I and if S is a subset of I , then $f|_S$ denotes the function f restricted to S . In this article, the symbols $C(I)$ and $AC(I)$ used in [1,12] are often represented by $C[a, b]$ and $AC[a, b]$ respectively. The following propositions 1 and 2 are fundamental in the present article:

Proposition 1. Let $a, b, c \in \mathbb{R}$ with $a < c < b$. The following statements are true:

(i) If $f \in C[a, b]$, $f|_{[a, c]} \in AC[a, c]$, $f|_{[c, b]} \in AC[c, b]$, then $f \in AC[a, b]$.

(ii) Suppose that $f \in C[a, b]$ is a monotone non-decreasing function. Let $h > 0$, let $d_h: [a, b - h] \rightarrow \mathbb{R}$ denote the function defined by $d_h(x) = f(x + h) - f(x)$. If d_h is a monotone non-decreasing function for all $h > 0$, then $f \in AC[a, b]$. If d_h is a monotone non-increasing function for all $h > 0$, then $f \in AC[a, b]$.

(iii) If $f \in AC[a, b]$ is a monotone non-decreasing function and if $g \in AC[f(a), f(b)]$, then $g \circ f \in AC[a, b]$.

Proof. The conclusions easily follow from the definition of absolutely continuous functions. //

Proposition 2. Let $g: [0, 1] \rightarrow [0, 1]$ be a continuous monotone increasing function defined by

$$g(x) = \sin^2(\pi x/2). \quad (2.1)$$

Then, the following statements are true:

(i) $g^{-1} \in AC[0, 1]$. (2.2)

(ii) For any $f \in AC[0, 1]$ there exist real numbers $\alpha(f)$ and $\beta(f)$ such that

$$\sum_{r=1}^N f \circ g(r/N) = \alpha(f)N + \beta(f) + o(1). \quad (2.3)$$

as $N \rightarrow \infty$.

Proof. (i) One easily verifies this by proposition 1(i-ii). (One can also easily verify (i) by 1(i) and the well-known fact that if $f \in C[a, b]$ is a convex function then, $f \in AC[a, b]$.)

(ii) Recall $\{K_N\} \in X_r(1)$ given by (2.2) in [9], and define $\{M_N\} \in X_r(1) \subset \mathcal{X}_{Hr}(1, 1)$ by

$$M_N = (1/4)K_N. \quad (2.4)$$

Let $f \in AC[0, 1]$. Notice that the j th eigenvalue $\lambda_j(M_N)$ of M_N , arranged in the increasing order, is given by

$$\lambda_j(M_N) = \sin^2((j-1)\pi/(2N)), \quad (2.5)$$

and that

$$\text{Tr } f(M_N) = \sum_{j=1}^N f(\lambda_j(M_N)) = \left[\sum_{r=1}^N f \circ g(r/N) \right] + f(0) - f(1). \quad (2.6)$$

By the Asymptotic Linearity Theorem (Practical ALT, $X_r(q)$ -version) reproduced below (cf. [11, 12] and references therein for details), the conclusion directly follows. //

Theorem PALT (Practical ALT, $X_r(q)$ -version). *Let $\{M_N\} \in X_r(q)$ be a fixed repeat sequence, let I be a fixed closed interval compatible with $\{M_N\}$. Then, for any $\varphi \in AC(I)$, there exist $\alpha(\varphi), \beta(\varphi) \in \mathbb{R}$ such that*

$$\text{Tr} \varphi(M_N) = \alpha(\varphi)N + \beta(\varphi) + o(1) \quad (2.7)$$

as $N \rightarrow \infty$.

Recall (1.2) and cf. (A.14) in the appendix for the definition of $X_r(q)$; a closed interval I is called compatible with $\{M_N\} \in X_r(q)$ if all the eigenvalues of M_N are contained in I for all $N \in \mathbb{Z}^+$ (cf. [12] and references therein for details). In proof (ii) of proposition 2 above, one can also apply the original version [9] of the ALT or the newest $\mathcal{X}_{\text{tr}}(q, 1)$ version [14] of the ALT to the sequence $\{M_N\}$.

The reader who is not familiar with the notion of the ‘function’ $\varphi(M)$ of an $n \times n$ normal matrix is referred to [12] and references therein. We shall recall here only the basic definition and property of $\varphi(M)$. Let

$$M = \mu_1 P_{(1)} + \dots + \mu_r P_{(r)} \quad (2.8)$$

be the spectral resolution of the normal matrix M , where $\mu_1, \dots, \mu_r$ are all the distinct eigenvalues of M and $P_{(1)}, \dots, P_{(r)}$ are corresponding eigenprojections. Let J be a subset of $\mathbb{C}$ that contains all the eigenvalues of M and let φ be a complex-valued function defined on J . Then, we define $\varphi(M)$ by

$$\varphi(M) = \varphi(\mu_1)P_{(1)} + \dots + \varphi(\mu_r)P_{(r)}. \quad (2.9)$$

The fact that it is well defined is easily seen by the uniqueness of the spectral resolution.

Let U be an $n \times n$ unitary matrix such that

$$M = U \text{diag}(\lambda_1, \dots, \lambda_n) U^{-1} \quad (2.10)$$

where $\lambda_1, \dots, \lambda_n$ are all the eigenvalues of M counted with multiplicity, then one gets

$$\varphi(M) = U \text{diag}(\varphi(\lambda_1), \dots, \varphi(\lambda_n)) U^{-1}. \quad (2.11)$$

The Matrix Art and Math Art Programs (using computer

graphic visualization of matrices) in the Fukui project are philosophical and methodical extensions, from science towards art, of Fukui’s approach and also of the Approach via the Aspect of Form and General Topology (cf. [1, 9, 12, 16] references therein) in the repeat space theory (RST), which is the fundamental unifying theory in the first and second generation Fukui project. In the above-mentioned programs, we often use special $n \times n$ matrices, which are referred to as central matrices. The definition of a central matrix is as follows:

Let $n \in \mathbb{Z}^+$, let $c = (c_1, c_2) \in \mathbb{R}^2$. An $n \times n$ complex matrix M is called a *central matrix with center c* in $\mathbb{R}^2$ if there exists a $p \in]0, \infty]$ and a function $f: [0, \infty[\rightarrow \mathbb{C}$ such that

$$M_{ij} = f(|i - c_1|^p + |j - c_2|^p)^{1/p} \quad (2.12)$$

for all $1 \leq i, j \leq n$, where

$$(|i - c_1|^\infty + |j - c_2|^\infty)^{1/\infty} := \max\{|i - c_1|, |j - c_2|\}. \quad (2.13)$$

The above central matrix M is denoted by

$$M = \text{Zen}(n, c, p, f). \quad (2.14)$$

(Remark: The German word ‘Zentrum’ means center.) Let $C^{[0, \infty[}$ denote the set of all functions $f: [0, \infty[\rightarrow \mathbb{C}$. Let $M_n(\mathbb{C})$ denote the complex linear space of all $n \times n$ complex matrices. One can also define the mapping

$$\text{Zen}: \mathbb{Z}^+ \times \mathbb{R}^2 \times]0, \infty] \times C^{[0, \infty[} \rightarrow \bigcup_{n \geq 1} M_n(\mathbb{C}) \quad (2.15)$$

by (2.12). We let Zen_n denote the set of all $n \times n$ central matrices. Thus, we have

$$\text{Zen}_n = \text{Zen}(\{n\} \times \mathbb{R}^2 \times]0, \infty] \times C^{[0, \infty[}). \quad (2.16)$$

Let span denote the linear span operation in $M_n(\mathbb{C})$. Then, it is easily seen that

$$\text{span } \text{Zen}_n = M_n(\mathbb{C}). \quad (2.17)$$

Let $n \in \mathbb{Z}^+$. An $n \times n$ complex matrix L is called a *circle matrix* if L is a central matrix with center $\gamma_n := ((n+1)/2, (n+1)/2)$. Thus, $n \times n$ circle matrix L is expressed in the form:

$$L = \text{Zen}(n, \gamma_n, p, f). \quad (2.18)$$

We let O_n denote the set of all $n \times n$ circle matrices. Thus, we have

$$O_n = \text{Zen}(\{n\} \times \{\gamma_n\} \times]0, \infty] \times C^{[0, \infty[}). \quad (2.19)$$

Matrix

$$M \in \text{span } O_n \quad (2.20)$$

is called a *span circle matrix*, or a *mandala matrix*.

(Mandala is a Sanskrit word that means circle. The basic form of most Hindu and Buddhist mandalas is a square containing circles and squares with a center point.)

An $n \times n$ complex matrix $M = A_1 + A_2 + \dots + A_k$, where $A_1, A_2, \dots, A_k \in \text{Zen}_n$, is said to be a k -centered matrix, if M cannot be expressed as a sum of k_0 central matrices with $k_0 < k$.

The following d -dimensional generalizations Zen^d and ζ^d of the mapping Zen have applications to quantum chemistry and physics.

Let $d \in \mathbb{Z}^+$ with $d \geq 2$. Define the mapping

$$\text{Zen}^d: \mathbb{Z}^+ \times \mathbb{R}^d \times]0, \infty] \times C^{[0, \infty[} \rightarrow \bigcup_{n \geq 1} M_n(\mathbb{C}) \quad (2.21)$$

by

$$\text{Zen}^d(n, c, p, f) = M(d, n, c, p, f), \quad (2.22)$$

where $M = M(d, n, c, p, f)$ with $c = (c_1, c_2, \dots, c_d) \in \mathbb{R}^d$ is an $n \times n$ complex matrix given by

$$M_{ij} = f((|i - c_1|^p + |j - c_2|^p)^{1/p}) \quad \text{if } d = 2;$$

$$M_{ij} = f((|i - c_1|^p + |j - c_2|^p + |0 - c_3|^p + \dots + |0 - c_d|^p)^{1/p}) \quad \text{if } d \geq 3; \quad (2.23)$$

for all $1 \leq i, j \leq n$. Here, the right sides of (2.23) with $p = \infty$ is given in the manner analogous to (2.13). Note that

$$\text{Zen} = \text{Zen}^2. \quad (2.24)$$

Let $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$ with $d \geq 2$, let

$$\text{Zen}_n^d := \text{Zen}^d(\{n\} \times \mathbb{R}^d \times]0, \infty] \times C^{[0, \infty[}). \quad (2.25)$$

Then, we have the relationship

$$\text{Zen}_n = \text{Zen}_n^2. \quad (2.26)$$

Let $O_n^2 := O_n$, and let

$$O_n^d := \text{Zen}(\{n\} \times (\{\gamma_n\} \times \mathbb{R}^{d-2}) \times]0, \infty] \times C^{[0, \infty[}) \quad (2.27)$$

if $d \geq 3$. An $n \times n$ complex matrix M is called a *central matrix centering in $\mathbb{R}^d$* if

$$M \in \text{Zen}_n^d, \quad (2.28)$$

a *circle matrix centering in $\mathbb{R}^d$* if

$$M \in O_n^d, \quad (2.29)$$

a *span circle matrix* or a *mandala matrix centering in $\mathbb{R}^d$* if

$$M \in \text{span } O_n^d. \quad (2.30)$$

Let $d \in \mathbb{Z}^+$ with $d \geq 2$. Let $C^{\mathbb{R}^d}$ denote the set of all functions $F: \mathbb{R}^d \rightarrow \mathbb{C}$. Define the mapping

$$\zeta^d: \mathbb{Z}^+ \times \mathbb{R}^d \times C^{\mathbb{R}^d} \rightarrow \bigcup_{n \geq 1} M_n(\mathbb{C}) \quad (2.31)$$

by

$$\zeta^d(n, c, F) = M(d, n, c, F), \quad (2.32)$$

where $M = M(d, n, c, F)$ with $c = (c_1, c_2, \dots, c_d) \in \mathbb{R}^d$ is an n

$\times n$ complex matrix given by

$$M_{ij} = F(i - c_1, j - c_2) \quad \text{if } d = 2;$$

$$M_{ij} = F(i - c_1, j - c_2, 0 - c_3, \dots, 0 - c_d)$$

$$\text{if } d \geq 3; \quad (2.33)$$

for all $1 \leq i, j \leq n$. Let $f \in C^{[0, \infty[}$, let $p \in]0, \infty]$, and

let $F_{p,f}: \mathbb{R}^d \rightarrow \mathbb{C}$ denote the function defined by

$$F_{p,f}(x_1, x_2, \dots, x_d) = f((|x_1|^p + |x_2|^p + \dots + |x_d|^p)^{1/p}). \quad (2.34)$$

Then, we have the relationship

$$\zeta^d(n, c, F_{p,f}) = \text{Zen}^d(n, c, p, f), \quad (2.35)$$

valid for all $(n, c, p, f) \in \mathbb{Z}^+ \times \mathbb{R}^d \times]0, \infty] \times C^{[0, \infty[}$.

Let $n \in \mathbb{Z}^+$ and $d \in \mathbb{Z}^+$ with $d \geq 2$, let

$$\zeta_n^d := \zeta^d(\{n\} \times \mathbb{R}^d \times C^{\mathbb{R}^d}). \quad (2.36)$$

An $n \times n$ complex matrix M is called the ζ matrix of index (d, n, c, F) if

$$M = \zeta^d(n, c, F). \quad (2.37)$$

A ζ matrix of index (d, n, c, F) is called a *vertical matrix* if

$$c_1 = c_2 = (n + 1)/2, \quad (2.38)$$

and if

$$F(x_1, x_2, \dots, x_d) = F(-x_1, x_2, \dots, x_d) \quad (2.39)$$

for all $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

A ζ matrix of index (d, n, c, F) is called a *horizontal matrix* if

$$c_1 = c_2 = (n + 1)/2, \quad (2.40)$$

and if

$$F(x_1, x_2, \dots, x_d) = F(x_1, -x_2, \dots, x_d) \quad (2.41)$$

for all $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

The set of all $n \times n$ vertical matrices is denoted by Ve_n and the set of all $n \times n$ horizontal matrices is denoted by Ho_n .

The patterns of the following matrices (i) and (ii) are analyzed in the fundamental part of the Matrix Art Program, in conjunction with the Approach via the Aspect of Form and General Topology in the repeat space theory (RST), which originally stemmed from the experimental and empirical soil of chemistry and engineering: (i) Real-symmetric vertical/horizontal matrices M and $\varphi(M)$ with real-valued functions φ defined on suitable domains. (ii) Sum $M = A_1 + A_2 + \dots + A_k$ of central matrices $A_1, A_2, \dots, A_k$, matrix B defined by $B_{ij} = \varphi_1(M_{ij}), \varphi_2(M^*M)$, and

$\varphi_3((1/2)(M^*+M))$, with real-valued functions $\varphi_1, \varphi_2, \varphi_3$ defined on suitable domains.

The development of the Fukui project along the above line and its implications shall be published elsewhere.

3. AFFIRMATIVE SOLUTION OF PROBLEM 1

In this section, the symbol $AC(I)$ denotes the Banach space of all real-valued absolutely continuous functions on I equipped with the norm given by

$$\|\varphi\| = \sup \{|\varphi(t)| : t \in I\} + V_I(\varphi),$$

where $V_I(\varphi)$ denotes the total variation of φ on I , i.e.,

$$V_I(\varphi) = \sup_A \sum_{i=1}^n |\varphi(t_i) - \varphi(t_{i-1})|.$$

$$(A: a = t_0 \leq t_1 \leq \dots \leq t_n = b)$$

First, we establish theorem 1 by using theorem 2. Second, we derive theorem 2 from the ALT.

Proof of theorem 1 by using theorem 2. We assume theorem 2. Under the assumption of theorem 2, we show that

$$\alpha(f) = (1/(b-a)) \left(\int_a^b f(\theta) d\theta \right), \quad (3.1)$$

and that

$$\beta(f) = (1/2)(f(b) - f(a)). \quad (3.2)$$

Since f is absolutely continuous it is Riemann integrable, hence dividing both sides of (1.3) by $N/(b-a)$ and letting $N \rightarrow \infty$, we see that (3.1) is true.

Now let $\beta_N: AC(I) \rightarrow \mathbb{R}$ denote the linear functionals defined by

$$\beta_N(f) = \sum_{r=1}^N f(x(N, r)) - (1/(b-a)) \left(\int_a^b f(\theta) d\theta \right) N, \quad (3.3)$$

$N \in \mathbb{Z}^+$.

Let $C(I)$ denote the subspace of $AC(I)$ of all continuously differentiable functions on I . By using Taylor's theorem, it is not difficult to show that for each $f \in C(I)$.

$$\beta_N(f) \rightarrow (1/2)(f(b) - f(a)) \quad (3.4)$$

as $N \rightarrow \infty$. (Or, by using the Euler-Maclaurin theorem, C' version, one immediately sees that (3.4) is true for all $f \in C(I)$.)

Define $\beta: AC(I) \rightarrow \mathbb{R}$ by

$$\beta(f) = \lim_{N \rightarrow \infty} \beta_N(f). \quad (3.5)$$

Note that this functional β is well defined in view of theorem 2. Recall the fact that $AC(I)$ is a Banach space (Cf. [11] and references therein). It is now immediately seen that β is a bounded linear functional by virtue of the

Banach-Steinhouse theorem (the Uniformly Boundedness theorem). Define $\beta^\#: AC(I) \rightarrow \mathbb{R}$ by

$$\beta^\#(f) = (1/2)(f(b) - f(a)). \quad (3.6)$$

Then, it is easily seen that $\beta^\#$ is a bounded linear functional. Note that

$$\beta(f) = \beta^\#(f) \quad (3.7)$$

for all $g \in C(I)$. Recall the fact that $C(I)$ is a dense subset of $AC(I)$:

$$\overline{C(I)} = AC(I). \quad (3.8)$$

By the continuity of β and $\beta^\#$, we see that $\beta(f) = \beta^\#(f)$ for all $g \in AC(I)$. Therefore

$$\beta = \beta^\#, \quad (3.9)$$

This completes the proof. //

Proof of theorem 2. By changing the variable, we may assume that $a = 0$, and $b = 1$. We have only to verify that if $f \in AC[0, 1]$ then there exist real numbers $\alpha(f)$ and $\beta(f)$ such that

$$\sum_{r=1}^N f(r/N) = \alpha(f)N + \beta(f) + o(1) \quad (3.10)$$

as $N \rightarrow \infty$. But, by proposition 2, which was established by using the ALT in section 2, there exists a continuous monotone increasing function $g: [0, 1] \rightarrow [0, 1]$ with $g^{-1} \in AC[0, 1]$ such that for any $f \in AC[0, 1]$ there exist real numbers $\alpha(f)$ and $\beta(f)$ such that

$$\sum_{r=1}^N f \circ g(r/N) = \alpha(f)N + \beta(f) + o(1) \quad (3.11)$$

as $N \rightarrow \infty$. Let $u \in AC[0, 1]$ be arbitrary. Recall proposition 1(iii) and note that

$$u \circ g^{-1} \in AC[0, 1]. \quad (3.12)$$

So, setting $f = u \circ g^{-1}$ in (3.11), we get the conclusion. //

Thus, the goal of the present article has been attained.

We remark that the notion of the normed repeat space reviewed in the appendix unites the approaches via the aspects of form and general topology exploited in a variety of asymptotic analyses of molecular networks in [1-18] and references therein. Equipped with the machinery of Banach algebras and C^* -algebras, the notion of normed repeat space with the above-mentioned new unifying power forms a basis of the second generation Fukui project. For a review of the first generation Fukui project, whose basic philosophy we would like to carry on to the second generation project, the reader is referred to ref. [2] entitled 'Note on the repeat space theory – its development and communications with Prof. Kenichi Fukui'.

4. APPENDIX

Review of the Generalized Repeat Space and the Normed Repeat Space

I. Generalized Repeat Space

There are several equivalent ways of defining the generalized repeat space $\mathcal{R}(q, d)$ with a given size $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. We shall recall below the definition that uses the notion of the sum of subspaces of a linear space (cf. Refs. [1,13,15,16]).

Fix $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $\mathcal{R}(q, d)$ denote the set of all matrix sequences whose N -th term M_N is an arbitrary $qN^d \times qN^d$ complex matrix, $N \in \mathbb{Z}^+$. This set constitutes a $*$ -algebra over the field $\mathbb{C}$ with term-wise addition, scalar multiplication, multiplication

$$\{M_N\} + \{M_N\} = \{M_N + M_N\}, \quad (\text{A.1})$$

$$k\{M_N\} = \{kM_N\}, \quad (\text{A.2})$$

$$\{M_N\} \{M_N\} = \{M_N M_N\}, \quad (\text{A.3})$$

and involution $(\cdot)^*$: $\mathcal{R}(q, d) \rightarrow \mathcal{R}(q, d)$ defined by

$$\{M_N\}^* = \{M_N^*\}, \quad (\text{A.4})$$

where the $*$ on the right-hand side of (A.4) denotes the adjoint operation.

Let P_N denote an $N \times N$ real-orthogonal matrix given by

$$P_N = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \mathbf{0} \\ & & 0 & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ \mathbf{0} & & & & & 0 & 1 \\ 1 & & & & & & 0 \end{pmatrix}.$$

Let $P_N^a := (P_N^{-1})^a$ where $a \in \{-2, -3, \dots\}$. (Note that P_N^a equals the transpose of P_N^{-a} .)

Let S_N denote an $N \times N$ real idempotent matrix given by

$$S_N = \begin{pmatrix} 1 & 0 & & & & \\ 0 & 0 & 0 & & & \mathbf{0} \\ & 0 & 0 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 0 \\ \mathbf{0} & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}.$$

Let P_N^n denote the $N^d \times N^d$ matrix given by

$$P_N^n = P_N^{n_1} \otimes P_N^{n_2} \otimes \dots \otimes P_N^{n_d}, \quad (\text{A.5})$$

where $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$, and $\otimes$ denotes the Kronecker product.

Let S_N^k denote the $N^d \times N^d$ matrix given by

$$S_N^k = S_N^{k_1} \otimes S_N^{k_2} \otimes \dots \otimes S_N^{k_d}, \quad (\text{A.6})$$

where $k = (k_1, k_2, \dots, k_d) \in (\mathbb{Z}^+ \cup \{0\})^d$.

Let $\mathcal{V}^k(q, d)$ with $k = (k_1, k_2, \dots, k_d) \in \{0, 1\}^d$ denote the subset of $\mathcal{R}(q, d)$ defined by

$$\mathcal{V}^k(q, d) = \{\{M_N\} \in \mathcal{R}(q, d) : \exists m, n \in \mathbb{Z}^d, \exists Q \in \mathbf{M}_q(\mathbb{C}) \text{ such that}$$

$$M_N = (P_N^m S_N^k P_N^n) \otimes Q \text{ for all } N \gg 0\}. \quad (\text{A.7})$$

Let $\text{span } \mathcal{V}^k(q, d)$ with $k = (k_1, k_2, \dots, k_d) \in \{0, 1\}^d$ denote the linear span of $\mathcal{V}^k(q, d)$.

We defined three fundamental linear subspaces

$\mathcal{R}(q, d)$, $\mathcal{R}_\alpha(q, d)$, and $\mathcal{R}_\beta(q, d)$ of $\mathcal{R}(q, d)$ by

$$\mathcal{R}(q, d) = \sum_{k \in \{0,1\}^d} \text{span } \mathcal{V}^k(q, d), \quad (\text{A.8})$$

$$\mathcal{R}_\alpha(q, d) = \text{span } \mathcal{V}^0(q, d), \quad (\text{A.9})$$

$$\text{where } \mathbf{0} = (0, 0, \dots, 0) \in \{0, 1\}^d, \quad (\text{A.10})$$

$$\mathcal{R}_\beta(q, d) = \sum_{k \in \{0,1\}^d \setminus \{\mathbf{0}\}} \text{span } \mathcal{V}^k(q, d). \quad (\text{A.11})$$

In (A.8) and (A.11), the Σ denotes the sum of subspaces in the obvious manner.

We call $\mathcal{R}(q, d)$, $\mathcal{R}_\alpha(q, d)$, $\mathcal{R}_\beta(q, d)$, respectively, the generalized repeat space, generalized alpha space, and generalized beta space with size (q, d) , and each element of $\mathcal{R}(q, d)$, $\mathcal{R}_\alpha(q, d)$, $\mathcal{R}_\beta(q, d)$, respectively, a generalized repeat sequence, generalized alpha sequence, and generalized beta sequence with size (q, d) .

The following is one of the most fundamental theorems in the repeat space theory.

Theorem A1. For all $q, d \in \mathbb{Z}^+$, $\mathcal{R}(q, d)$ forms a $*$ -algebra.

Proof. This was proved in Ref. [15]. //

For the special definition of the generalized repeat space with size $(q, 1)$, set $d=1$ in the definition of $\mathcal{V}^k(q, d)$ given by (A.7) and observe that

$$\mathcal{R}_\alpha(q, 1) = \text{span } \mathcal{V}^0(q, 1)$$

= $\text{span } \{\{M_N\} \in \mathcal{R}(q, 1) : \exists m \in \mathbb{Z}, \exists Q \in \mathbf{M}_q(\mathbb{C}) \text{ such that}$

$$M_N = P_N^m \otimes Q$$

$$\text{for all } N \gg 0\}, \quad (\text{A.12})$$

$$\mathcal{R}_\beta(q, 1) = \text{span } \mathcal{V}^1(q, 1)$$

= $\text{span } \{\{M_N\} \in \mathcal{R}(q, 1) : \exists m, n \in \mathbb{Z}, \exists Q \in \mathbf{M}_q(\mathbb{C}) \text{ such that}$

$$M_N = (P_N^m S_N P_N^n) \otimes Q$$

$$\text{for all } N \gg 0\}, \quad (\text{A.13})$$

and note that

$$\mathcal{R}(q, 1) = \mathcal{R}_\alpha(q, 1) + \mathcal{R}_\beta(q, 1). \quad (\text{A.14})$$

II. Normed Repeat Space

Let $\mathbf{C}^n$ denote the set of all column n -vectors. For each $1 \leq p < \infty$, let

$$\|\xi\|_p := \|(\xi_1, \dots, \xi_n)^T\|_p = (|\xi_1|^p + \dots + |\xi_n|^p)^{1/p}. \quad (\text{B.1})$$

Let

$$\|\xi\|_\infty = \|(\xi_1, \dots, \xi_n)^T\|_\infty = \max \{|\xi_i| : 1 \leq i \leq n\}. \quad (\text{B.2})$$

For each positive integer n and $1 \leq p < \infty$, let $\text{Mat}(n, p)$ denote the set of all $n \times n$ complex matrices with the norm given by

$$\|A\|_p = \sup \{ \|Ax\|_p / \|x\|_p : x \in \mathbf{C}^n - \{0\} \}. \quad (\text{B.3})$$

Fix $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $\mathcal{A}(q, d)$ denote the set of all matrix sequences whose N -th term M_N is an arbitrary $qN^d \times qN^d$ complex matrix, $N \in \mathbb{Z}^+$. This set constitutes a $*$ -algebra over the field $\mathbb{C}$ with term-wise addition, scalar multiplication, multiplication

$$\{M_N\} + \{M_N\} = \{M_N + M_N\}, \quad (\text{B.4})$$

$$k\{M_N\} = \{kM_N\}, \quad (\text{B.5})$$

$$\{M_N\}\{M_N\} = \{M_N M_N\}, \quad (\text{B.6})$$

and involution $(\cdot)^*$: $\mathcal{A}(q, d) \rightarrow \mathcal{A}(q, d)$ defined by

$$\{M_N\}^* = \{M_N^*\}, \quad (\text{B.7})$$

where the $*$ on the right-hand side of (B.7) denotes the adjoint operation.

For each $q, d \in \mathbb{Z}^+$ and $1 \leq p < \infty$, let

$$\begin{aligned} \mathcal{B}(q, d, p) &:= \\ \{ \{M_N\} \in \prod_{N=1}^{\infty} \text{Mat}(qN^d, p) : \|\{M_N\}\|_p &:= \\ \sup_N \|M_N\|_p < \infty \}. \end{aligned} \quad (\text{B.8})$$

Note that $\mathcal{B}(q, d, p)$ forms a subalgebra of $\mathcal{A}(q, d)$. We also note that $\mathcal{B}(q, d, p)$ forms a Banach algebra for each $1 \leq p < \infty$ and a C^* -algebra for $p = 2$. The set $\mathcal{B}(q, d, p)$ is called the *bounded underlying space* (or *B-space* for short) of type (q, d, p) .

Now recall the definition of the generalized repeat space with size (q, d) , which is denoted by $\mathcal{A}(q, d)$ in (A.8).

Theorem B1. For each $q, d \in \mathbb{Z}^+$ and $1 \leq p < \infty$, we have

$$\mathcal{A}(q, d) \subset \mathcal{B}(q, d, p). \quad (\text{B.9})$$

Proof. This was proved in Ref. [3]. //

Definition of the Normed Repeat Space For each $q, d \in \mathbb{Z}^+$ and $1 \leq p < \infty$, let

$$\mathcal{R}(q, d, p) := \text{closure of } \mathcal{A}(q, d) \subset \mathcal{B}(q, d, p). \quad (\text{B.10})$$

The set $\mathcal{R}(q, d, p)$ is called the *normed repeat space* of type (q, d, p) .

Note that $\mathcal{R}(q, d, p)$ forms a Banach algebra for each $1 \leq p < \infty$ and a C^* -algebra for $p = 2$. This fact easily follows

from the observation that linear operations, multiplication, and involution are all continuous operations and that any closed set in a complete metric space forms a complete metric subspace. (The reader is referred e.g. to refs. [19,20] for the fundamental properties of Banach algebras and C^* -algebras.)

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